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CYCLICALLY COMPACT OPERATORS IN BANACH SPACES

A. G. Kusraev

The Boolean-valued interpretation of compactness gives rise to the new notions of cyclically compact sets and operators which deserves an independent study. A part of the corresponding theory is presented in this work. General form of cyclically compact operators in Kaplansky–Hilbert module as well as a variant of Fredholm Alternative for cyclically compact operators are also given.

1. Preliminaries

In this section we present briefly some basic facts about Boolean-valued representations which we need in the sequel.

1.1. Let B be a complete Boolean algebra and let A be a nonempty set. Recall (see [3]) that $B(\mathrm{A})$ denotes the set of all partitions of unity in B with the fixed index set A. More precicely, assign

$$B(A) := \Big\{\nu : A \to B : (\forall \alpha, \beta \in A)\ \big(\alpha \neq \beta \to \nu(\alpha) \wedge \nu(\beta) = 0\big) \\ \wedge \bigvee\nolimits_{\alpha \in A} \nu(\alpha) = 1\Big\}.$$

If A is an ordered set then we may order the set $B(A)$ as well:

$$\nu \ \leq \ \mu \leftrightarrow (\forall \alpha, \beta \in A)\ \big(\nu(\alpha) \wedge \mu(\beta) \neq 0 \to \alpha \leq \beta\big) \quad \big(\nu, \mu \in B(A)\big).$$

It is easy to show that this relation is actually a partial order in $B(A)$. If A is directed upward (downward) then so does $B(A)$. Let Q be the Stone space of the algebra B. Identifying an element $\nu(\alpha)$ with a clopen subset of Q, we construct the mapping $\bar{\nu} : Q_\nu \to A$, $Q_\nu := \bigcup\big\{\nu(\alpha) : \alpha \in A\big\}$, by letting $\bar{\nu}(q) = \alpha$ whenever $q \in \nu(\alpha)$. Thus, $\bar{\nu}$ is a step-function that takes the value α on $\nu(\alpha)$. Moreover, $\nu \leq \mu \to (\forall q \in Q_\nu \cap Q_\mu)\ \big(\bar{\nu}(q) \leq \bar{\mu}(q)\big)$.

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1.2. Let X be a normed space. Suppose that $\mathcal{L}(X)$ has a complete Boolean algebra of norm one projections $\mathcal{B}$ which is isomorphic to B. In this event we will identify the Boolean algebras $\mathcal{B}$ and B, writing $B \subset \mathcal{L}(X)$. Say that X is a *normed B-space* if $B \subset \mathcal{L}(X)$ and for every partition of unity $(b_\xi)_{\xi\in\Xi}$ in B the two conditions hold:

(1) If $b_\xi x = 0$ $(\xi \in \Xi)$ for some $x \in X$ then $x = 0$;

(2) If $b_\xi x = b_\xi x_\xi$ $(\xi \in \Xi)$ for $x \in X$ and a family $(x_\xi)_{\xi\in\Xi}$ in X then $\|x\| \leq \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

Conditions (1) and (2) amount to the respective conditions ($1'$) and ($2'$):

($1'$) To each $x \in X$ there corresponds the greatest projection $b \in B$ such that $bx = 0$;

($2'$) If x, (x_ξ), and (b_ξ) are the same as in (2) then $\|x\| = \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

From ($2'$) it follows in particular that

$$\left\|\sum_{k=1}^{n} b_k x\right\| = \max_{k=1,\dots,n} \|b_k x\|$$

for $x \in X$ and pairwise disjoint projections $b_1, \dots, b_n$ in B.

Given a partition of unity (b_ξ), we refer to $x \in X$ satisfying the condition $(\forall\, \xi \in \Xi)\, b_\xi x = b_\xi x_\xi$ as a *mixing* of (x_ξ) by (b_ξ). If (1) holds then there is a unique mixing x of (x_ξ) by (b_ξ). In these circumstances we naturally call x the *mixing* of (x_ξ) by (b_ξ). Condition (2) maybe paraphrased as follows: The unit ball U_X of X is closed under mixing or is *mix-complete*.

1.3. Consider a normed B-space X and a net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in it. For every $\nu \in B(\mathrm{A})$ put $x_\nu := \operatorname{mix}_{\alpha\in\mathrm{A}}(\nu(\alpha)x_\alpha)$. If all the mixings exist then we come to a new net $(x_\nu)_{\nu\in B(\mathrm{A})}$ in X. Every subnet of the net $(x_\nu)_{\nu\in B(\mathrm{A})}$ is called a *cyclical subnet* of the original net $(x_\alpha)_{\alpha\in\mathrm{A}}$. If $s : \mathrm{A} \to X$ and $\varkappa : \mathrm{A}' \to B(\mathrm{A})$ then the mapping $s \bullet \varkappa : \mathrm{A}' \to X$ is defined by $s \bullet \varkappa(\alpha) := x_\nu$ where $\nu = \varkappa(\alpha)$. A *cyclical subsequence* of a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ is a sequence of the form $(x_{\nu_k})_{k\in\mathbb{N}}$ where $(\nu_k)_{k\in\mathbb{N}}$ is a sequence in $B(\mathbb{N})$ with $\nu_k \ll \nu_{k+1}$ for all $k \in \mathbb{N}$.

1.4. Let Λ be the bounded part of the universally complete K-space $\mathcal{C}{\downarrow}$, i. e. Λ is the order-dense ideal in $\mathcal{C}{\downarrow}$ generated by the order-unity $\mathbf{1} := 1^\wedge \in \mathcal{C}{\downarrow}$. Take a Banach space $\mathcal{X}$ inside $\mathbf{V}^{(B)}$. Denote (see [1])

$$\mathcal{X}{\downarrow}^\infty := \{x \in \mathcal{X}{\downarrow} : |x| \in \Lambda\}.$$

Then $\mathcal{X}{\downarrow}^\infty$ is a Banach–Kantorovich space called the *bounded descent* of $\mathcal{X}$. Since Λ is an order complete AM-space with unity, $\mathcal{X}{\downarrow}^\infty$ is a Banach space with mixed norm over Λ. If $\mathcal{Y}$ is another Banach space and $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator inside $\mathbf{V}^{(B)}$ with $|\mathcal{T}{\downarrow}| \in \Lambda$ then the bounded descent of $\mathcal{T}$ is the restriction of $\mathcal{T}{\downarrow}$ to $\mathcal{X}{\downarrow}^\infty$. Clearly, the bounded descent of $\mathcal{T}$ is a bounded linear operator from $\mathcal{X}{\downarrow}^\infty$ to $\mathcal{Y}{\downarrow}^\infty$.

1.5. A normed B-space X is B-*cyclic* if we may find in X a mixing of each norm-bounded family by any partition of unity in B.

Theorem. *A Banach space X is linearly isometric to the bounded descent of some Banach space inside $\mathbf{V}^{(B)}$ if and only if X is B-cyclic.*

According to above theorem there is no loss of generality in assuming that X is a decomposable subspace of the Banach–Kantorovich space $\mathcal{X}{\downarrow}$, where $\mathcal{X}$ is a Banach space inside $\mathbf{V}^{(B)}$ and every projection $b \in B$ coincides with the restriction of $\chi(b)$ onto X. More precisely, we will assume that X is the bounded descent of $\mathcal{X}$, i.e., $X = \{x \in \mathcal{X}{\downarrow} \colon |x| \in \Lambda\}$, where Λ is the Stone algebra $\mathcal{S}(B)$ identified with the bounded part of the complex algebra $\mathcal{C}{\downarrow}$. In this event a subset $C \subset X$ is mix-complete if and only if $C = C{\uparrow}{\downarrow}$.

1.6. *Given a sequence $\sigma : \mathbb{N}^\wedge \to C{\uparrow}$ and $\varkappa : \mathbb{N}^\wedge \to \mathbb{N}^\wedge$, the composite $\sigma \downarrow \circ \varkappa{\downarrow}$ is a cyclical subsequence of the sequence $\sigma{\downarrow} : \mathbb{N} \to C$ if and only if $[\![\, \sigma \circ \varkappa$ is a subsequence of $\sigma \,]\!] = \mathbf{1}$. Given a sequence $s : \mathbb{N} \to C$ and $\varkappa : \mathbb{N} \to B(\mathbb{N})$, the composite $s{\uparrow} \circ \varkappa^\wedge$ is a subsequence of the sequence $\sigma{\uparrow} : \mathbb{N}^\wedge \to C{\uparrow}$ inside $\mathbf{V}^(B)$ if and only if $s \bullet \varkappa$ is a cyclical subsequence of the sequence s.*

2. Cyclically compact sets and operators

In this section we introduce cyclically compact sets and operators and consider some of their properties.

2.1. A subset $C \in X$ is said to be *cyclically compact* if C is mix-complete (see 1.5) and every sequence in C has a cyclic subsequence that converges (in norm) to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set.

A set $C \subset X$ is cyclically compact (relatively cyclically compact) if and only if $C{\uparrow}$ is compact (relatively compact) in $\mathcal{X}$.

$\lhd$ It suffices to prove the claim about cyclical compactness. In view of [1; Theorem 5.4.2] we may assume that $X = \mathcal{X}{\downarrow}$. Suppose that $[\![\, C{\uparrow}$ is compact $]\!] = \mathbf{1}$. Take an arbitrary sequence $s : \mathbb{N} \to C$. Then $[\![\, s{\uparrow} : \mathbb{N}^\wedge \to C{\uparrow}$ is a sequence in $C{\uparrow} \,]\!] = \mathbf{1}$. By assumption $C{\uparrow}$ is compact inside $\mathbf{V}^{(B)}$, so that there exist $\rho, x \in \mathbf{V}^{(B)}$ with $[\![\, \rho$ is a subsequence of $s{\uparrow} \,]\!] = [\![\, x \in C{\uparrow} \,]\!] = [\![\, \lim(\rho) = x \,]\!] = \mathbf{1}$. Since C is mix-complete, we obtain that $\rho{\downarrow}$ is a cyclical subsequence of s and $\lim(\rho\downarrow) = x \in C$. Conversely, suppose that C is a cyclically compact set. Take a sequence $\sigma : \mathbb{N}^\wedge \to C{\uparrow}$ in C. By assumption the sequence $\sigma{\downarrow} : \mathbb{N} \to C$ has a cyclic subsequence $\rho : B(\mathbb{N}) \to C$ converging to some $x \in C$. It remains to observe that $[\![\, \rho{\uparrow}$ is a subsequence of the sequence $\sigma \,]\!] = \mathbf{1}$ and $[\![\, \lim(\rho{\uparrow}) = x \,]\!] = \mathbf{1}$. $\rhd$

2.2. Theorem. *A mix-complete set C in a Banach B-space X is relatively cyclically compact if and only if for every $\varepsilon > 0$ there exist a countable partition of unity (π_n) in the Boolean algebra $\mathfrak{B}(X)$ and a sequence (θ_n) of finite subsets $\theta_n \subset C$ such that the set $\pi_n\big(\operatorname{mix}(\theta_n)\big)$ is an ε-net for $\pi_n(C)$ for all $n \in \mathbb{N}$. The last means that if*

$$\theta_n := \{x_{n,1}, \dots, x_{n,l(n)}\}$$

then for every $x \in \pi_n(C)$ there exists a partition of unity $\{\rho_{n,1}, \dots, \rho_{n,l(n)}\}$ in $\mathfrak{B}(X)$ with

$$\left\| x \Leftrightarrow \sum_{k=1}^{l(n)} \pi_n \rho_{n,k} x_{n,k} \right\| \le \varepsilon.$$

$\lhd$ According to 1.5 we may assume that $X := \mathcal{X}{\downarrow}$ for some Banach space $\mathcal{X}$ inside $\mathbf{V}^{(B)}$. By 2.1 a set $C \subset X$ is relatively cyclically compact if and only if $[\![\, C{\uparrow}$ is relatively compact $]\!] = \mathbf{1}$. By applying the Hausdorff Criterion to $C{\uparrow}$ inside $\mathbf{V}^{(B)}$, we obtain that relative cyclical compactness of $C{\uparrow}$ is equivalent to $[\![\, C{\uparrow}$ is totally bounded $]\!] = \mathbf{1}$ or, what amounts to the same, the following formula is valid inside $\mathbf{V}^{(B)}$:

$$(\forall 0 < \varepsilon \in \mathbb{R}^{\wedge})\, (\exists n \in \mathbb{N}^{\wedge})\, (\exists f : n \to \mathcal{X})\, (\forall x \in C{\uparrow})\, (\exists k \in n)$$
$$(\|x \Leftrightarrow f(k)\| \le \varepsilon).$$

Writing out Boolean truth values for the quantifiers, we see that the last claim can be stated in the following equivalent form: for every $0 < \varepsilon \in \mathbb{R}$ there exist a countable partition of unity (b_n) in B and a sequence (f_n) of elements of $\mathbf{V}^{(B)}$ such that $[\![\, f_n : n^{\wedge} \to \mathcal{X} \,]\!] \ge b_n$ and

$$[\![\, (\forall x \in C{\uparrow})\, (\exists k \in n^{\wedge})(\|x \Leftrightarrow f_n(k)\| \le \varepsilon^{\wedge}) \,]\!] \ge b_n.$$

Substitute f_n for $\operatorname{mix}(b_n f_n, b_n^* g_n)$, where g_n is an element of $\mathbf{V}^{(B)}$ with $[\![\, g_n : n^{\wedge} \to \mathcal{X} \,]\!] = \mathbf{1}$. Then f_n meets the above properties and obeys the additional requirement $[\![\, f_n : n^{\wedge} \to \mathcal{X} \,]\!] = \mathbf{1}$. Denote $h_n := f_n{\downarrow}$. So, the above implies that for every $x \in C$ holds

$$\bigvee \{ [\![\, \|x \Leftrightarrow h_n(k)\| \le \varepsilon^{\wedge} \,]\!] : \; k \in n \} \ge b_n.$$

Let $\chi : B \to \mathfrak{B}(X)$ be the isomorphism from 1.5 and put $\pi_k := \chi(b_k)$. If $b_{n,k} := [\![\, \|x \Leftrightarrow h_n(k)\| \le \varepsilon^{\wedge} \,]\!]$ and $x' := \sum_{k=0}^{n-1} j(b_{n,k}) h_n(k)$ then $[\![\, \|x' \Leftrightarrow x\| \le \varepsilon^{\wedge} \,]\!] = \mathbf{1}$, or equivalently $|\pi_n(x \Leftrightarrow x')| \le \varepsilon \mathbf{1}$. Thus, putting $\theta_n := \{h_n(0), \dots, h_n(n \Leftrightarrow 1)\}$, we obtain the desired sequence θ_n of finite subsets of C. $\rhd$

2.3. Denote by $\mathcal{L}_B(X, Y)$ the set of all bounded B-linear operators from X to Y. In this event $W := \mathcal{L}_B(X, Y)$ is a Banach space and $B \subset W$. If Y is B-cyclic then so is W. A projection $b \in B$ acts in W by the rule $T \mapsto b \circ T$ $(T \in W)$. We call $X^{\#} := \mathcal{L}_B(X, \Lambda)$ the *B-dual* of X. For every $f \in X^{\#}$ define a seminorm p_f on X by $p_f : x \mapsto \|f(x)\|_\infty$ $(x \in X)$. Denote by $\sigma_\infty(X, X^{\#})$ the topology in X generated by the family of seminorms $\{p_f : f \in X^{\#}\}$.

A mix-complete convex set $C \subset X$ is cyclically $\sigma_\infty(X, X^{\#})$-compact if and only if $C{\uparrow}$ is $\sigma(\mathcal{X}, \mathcal{X}^*)$-compact inside $\mathbf{V}^{(B)}$.

$\lhd$ The algebraic part of the claim is easy. Let the formula $\psi(\mathcal{A}, u)$ formalize the sentence: u belongs to the weak closure of $\mathcal{A}$. Then the formula can be written as

$$(\forall n \in \mathbb{N})\,(\forall \theta \in \mathcal{P}_{\mathrm{fin}}(\mathcal{X}))\,(\exists v \in \mathcal{A})\,(\forall y \in \theta)\,|(x \mid y)| \le n^{-1},$$

where ω is the set of naturals, $(\cdot \mid \cdot)$ is the inner product in $\mathcal{X}$, and $\mathcal{P}_{\mathrm{fin}}(\mathcal{X})$ is the set of all finite subsets of X. Suppose that $[\![\psi(\mathcal{A}, u)]\!] = \mathbf{1}$. Observe that

$$\mathcal{P}_{\mathrm{fin}}(X{\uparrow}) = \{\theta{\uparrow} \colon \theta \in \mathcal{P}_{\mathrm{fin}}(X)\}{\uparrow}$$

Using the Maximum Principle and the above relation, we may calculate Boolean truth values and arrive at the following assertion: For any $n \in \omega$ and any finite collections $\theta := \{y_1, \dots, y_m\}$ in $X^{\#}$, there exists $v \in \mathcal{A}{\downarrow}$ such that

$$[\![(\forall y \in \theta^{\wedge})\,|(u \Leftrightarrow v \mid y)| \le 1/n^{\wedge}]\!] = \mathbf{1}.$$

Moreover, we may choose v so that the extra condition $[\![\|v\| \le |u|]\!] = \mathbf{1}$ holds. Therefore,

$$|v| \le |u|, \quad |\langle (u \Leftrightarrow v) \mid y_l \rangle| < n^{-1}\mathbf{1} \quad (k := 1, \dots, n;\ l := 1, \dots, m).$$

There exists a fixed partition of unity $(e_\xi)_{\xi \in \Xi} \subset B$ which depends only on u and is such that $e_\xi |u| \in \Lambda$ for all ξ. From here it is seen that $e_\xi u \in A$ and $e_\xi v \in A$. Moreover,

$$\|\langle e_\xi (u \Leftrightarrow v) \mid y_l \rangle\|_\infty < n^{-1} \quad (k := 1, \dots, n;\ l := 1, \dots, m).$$

Repeating the above argument in the opposite direction, we come to the following conclusion: The formula $\psi(\mathcal{A}, u)$ is true inside $\mathbf{V}^{(B)}$ if and only if there exist a partition of unity $(e_\xi)_{\xi \in \Xi}$ in B and a family $(u_\xi)_{\xi \in \Xi}$ such that u_ξ belongs to the σ_∞-closure of A and $u = \operatorname{mix}(e_\xi u_\xi)$.

Now, assume that A is σ_∞-closed and the formula $\psi(\mathcal{A}, u)$ is true inside $\mathbf{V}^{(B)}$. Then u_ξ is contained in A by assumption and $[\![u_\xi \in \mathcal{A}]\!] = \mathbf{1}$. Hence $e_\xi \le [\![u \in \mathcal{A}]\!]$ for all ξ, i.e., $[\![u \in \mathcal{A}]\!] = \mathbf{1}$. Therefore,

$$\mathbf{V}^{(B)} \models (\forall u \in \mathcal{L}(\mathcal{X}))\psi(\mathcal{A}, u) \to u \in \mathcal{A}.$$

Conversely, assume $\mathcal{A}$ to be weakly closed. If u belongs to the σ_∞-closure of A, then u is contained in the weak closure of $\mathcal{A}$. $\triangleright$

2.4. Consider $X^{\#\#} := (X^{\#})^{\#} := \mathcal{L}_B(X^{\#}, \Lambda)$, the *second B-dual* of X. Given $x \in X$ and $f \in X^{\#}$, put $x^{\#\#} := \iota(x)$ where $\iota(x) : f \mapsto f(x)$. Undoubtedly, $\iota(x) \in L(X^{\#}, \Lambda)$. In addition,

$$\lvert x^{\#\#}\rvert = \lvert\iota(x)\rvert = \sup\{\lvert\iota(x)(f)\rvert : \lvert f\rvert \le \mathbf{1}\}$$
$$= \sup\{\lvert f(x)\rvert : (\forall x \in X)\lvert f(x)\rvert \le \lvert x\rvert\} = \sup\{\lvert f(x)\rvert : f \in \partial(\lvert\cdot\rvert)\} = \lvert x\rvert.$$

Thus, $\iota(x) \in X^{\#\#}$ for every $x \in X$. It is evident that the operator $\iota : X \to X^{\#\#}$, defined as $\iota : x \mapsto \iota(x)$, is linear and isometric. The operator ι is referred to as the *canonical embedding* of X into the second B-dual. As in the case of Banach spaces, it is convenient to treat x and $x^{\#\#} := \iota x$ as the same element and consider X as a subspace of $X^{\#\#}$. A B-normed space X is said to be *B-reflexive* if X and $X^{\#\#}$ coincide under the indicated embedding ι.

Theorem. *A normed B-space is B-reflexive if and only if its unit ball is cyclically $\sigma_\infty(X, X^{\#})$-compact.*

$\triangleleft$ The Kakutani Criterion claims that a normed space is reflexive if and only if its unit ball is weakly compact. Hence, the result follows from 2.3. $\triangleright$

2.5. Let X and Y be normed B-spaces. An operator $T \in \mathcal{L}_B(X, Y)$, is called *cyclically compact* (in symbols, $T \in \mathcal{K}_B(X, Y)$) if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact in Y. It is easy to see that $\mathcal{K}_B(X, Y)$ is a decomposable subspace of the Banach–Kantorovich space $\mathcal{L}_B(X, Y)$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Boolean-valued representations of X and Y. Recall that the immersion mapping $T \mapsto T^\sim$ of the operators is a linear isometric embedding of the lattice-normed spaces $\mathcal{L}_B(X, Y)$ into $\mathcal{L}^B(\mathcal{X}, \mathcal{Y})\!\downarrow$, see [1; Theorem 5.5.9]. Assume that Y is a B-cyclic space.

(1) *A bounded operator T from X into Y is cyclically compact if and only if $[\![\, T^\sim$ is a compact operator from $\mathcal{X}$ into $\mathcal{Y}\,]\!] = \mathbf{1}$.*

$\triangleleft$ Observe that C is bounded in X if and only if $[\![\, C^\sim$ is bounded in $\mathcal{X}\,]\!] = 1$. Moreover, according to [1: 3.4.14],

$$\mathbf{V}^{(B)} \models T(C)^\sim = T^\sim(C^\sim).$$

It remains to apply 2.1. $\triangleright$

(2) *$\mathcal{K}_B(X, Y)$ is a bo-closed decomposable subspace in $\mathcal{L}_B(X, Y)$.*

$\triangleleft$ Let $\mathcal{X}$ and $\mathcal{Y} \in \mathbf{V}^{(B)}$ be the same as above and let $\mathcal{K}^{(B)}(\mathcal{X}, \mathcal{Y})$ be the space of compact operators from $\mathcal{X}$ into $\mathcal{Y}$ inside $\mathbf{V}^{(B)}$. As was shown in [1; Theorem 5.9.9 the mapping $T \to T^\sim$ is an isometric embedding of $\mathcal{L}_B(X, Y)$ into $\mathcal{L}^{(B)}(\mathcal{X}, \mathcal{Y})\!\downarrow$.

It follows from (1) that this embedding maps the subspace $\mathcal{K}_B(X,Y)$ onto the bounded part of $\mathcal{K}^{(B)}(\mathcal{X},\mathcal{Y}){\downarrow}$. Taking into consideration the ZFC-theorem claiming the closure of the subspace of compact operators, we have $[\![\, \mathcal{K}^{(B)}(\mathcal{X},\mathcal{Y})$ is a closed subspace in

$$\mathcal{L}^{(B)}(\mathcal{X},\mathcal{Y})\,]\!] = \mathbf{1}.$$

From this we deduce that $\mathcal{K}^{(B)}(\mathcal{X},\mathcal{Y})\downarrow$ is bo-closed and decomposable in $\mathcal{L}^{(B)}(\mathcal{X},\mathcal{Y}){\downarrow}$. Thus, the bounded part of $\mathcal{K}^{(B)}(\mathcal{X},\mathcal{Y}){\downarrow}$ is also bo-closed and decomposable. $\triangleright$

(3) *Let $T \in \mathcal{L}_B(X,Y)$ and $S \in \mathcal{L}_B(Y,Z)$. If either T or S is cyclically compact then $S \circ T$ is also cyclically compact.*

$\triangleleft$ We need only to immerse the composite $S \circ T$ inside $\mathbf{V}^{(B)}$ and, taking into account (1) and [1; 3.4.14], apply therein the ZFC-theorem about compactness of the composite of a bounded operator and a compact operator. The subsequent descent leads immediately to the desired result. $\triangleright$

(4) *A bounded operator T is cyclically compact if and only if its adjoint T^* is cyclically compact.*

$\triangleleft$ Apply the above procedure, immersion into a Boolean-valued model and the subsequent descent. Observe that the operator $(T^*)^\sim$ is the adjoint of $T^\sim$ inside $\mathbf{V}^{(B)}$ and use the corresponding ZFC-theorem on compactness of the adjoint of a compact operator. $\triangleright$

3. Cyclically compact operators in Kaplansky–Hilbert modules

Now we consider general form of cyclically compact operators in Kaplansky–Hilbert modules.

3.1. Let Λ be a Stone algebra and consider a unitary Λ-module X. The mapping $\langle\cdot\,|\,\cdot\rangle : X \times X \to \Lambda$ is a Λ-*valued inner product*, if for all $x, y, z \in X$ and $a \in \Lambda$ the following are satisfied:

(1) $\langle x\,|\,x\rangle \geq \mathbf{0}$; $\langle x\,|\,x\rangle = \mathbf{0} \Leftrightarrow x = \mathbf{0}$;

(2) $\langle x\,|\,y\rangle = \langle y\,|\,x\rangle^*$;

(3) $\langle ax\,|\,y\rangle = a\langle x\,|\,y\rangle$;

(4) $\langle x + y\,|\,z\rangle = \langle x\,|\,z\rangle + \langle y\,|\,z\rangle$.

Using a Λ-valued inner product, we may introduce the norm in X by the formula

(5) $|\!|\!|x|\!|\!| := \sqrt{\|\langle x|x\rangle\|} \quad (x \in X),$

and the vector norm

(6) $\lvert x\rvert := \sqrt{\langle x|x\rangle} \quad (x \in X)$.

3.2. Let X be a Λ-module with an inner product $\langle\cdot\,|\,\cdot\rangle : X \times X \to \Lambda$. If X is complete with respect to the mixed norm $|||\cdot|||$, it is called a *C^*-module* over Λ. A *Kaplansky–Hilbert module* or an *AW^*-module* over Λ is a unitary C^*-module over Λ that enjoys the following two properties:

(1) let x be an arbitrary element in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$ with $e_\xi x = 0$ for all $\xi \in \Xi$; then $x = 0$;

(2) let $(x_\xi)_{\xi\in\Xi}$ be a norm-bounded family in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$; then there exists an element $x \in X$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$.

The element of (2) is the *bo*-sum of the family $(e_\xi x_\xi)_{\xi\in\Xi}$. According to the Cauchy–Bunyakovskiĭ–Schwarz inequality $\langle x\,|\,y\rangle \le \lvert x\rvert\lvert y\rvert$ the inner product is *bo*-continuous in each variable. In particular,

If X is a C^*-module than the pair $(X, |||\cdot|||)$ is a B-cyclic Banach space if and only if $(X, \lvert\cdot\rvert)$ is a Banach–Kantorovich space over $\Lambda := \mathcal{S}(B)$, see [1; Theorem 6.2.7].

3.4. Theorem. *The bounded descent of an arbitrary Hilbert space in $\mathbf{V}^{(B)}$ is a Kaplansky–Hilbert module over the Stone algebra $\mathcal{S}(B)$. Conversely, if X is a Kaplansky–Hilbert module over $\mathcal{S}(B)$, then there is a Hilbert space $\mathcal{X}$ in $\mathbf{V}^{(B)}$ whose bounded descent is unitarily equivalent with X. This space is unique to within unitary equivalence inside $\mathbf{V}^{(B)}$.*

$\lhd$ The proof can be found in [1; Theorem 6.2.8] $\rhd$

3.5. Theorem. *Let T in $\mathcal{K}_B(X,Y)$ be a cyclically compact operator from a Kaplansky–Hilbert module X to a Kaplansky–Hilbert module Y. There are orthonormal families $(e_k)_{k\in\mathbb{N}}$ in X, $(f_k)_{k\in\mathbb{N}}$ in Y, and a family $(\mu_k)_{k\in\mathbb{N}}$ in Λ such that the following hold:*

(1) *$\mu_{k+1} \le \mu_k$ $(k \in \mathbb{N})$ and $o\text{-}\lim_{k\to\infty} \mu_k = 0$;*

(2) *there exists a projection π_∞ in Λ such that $\pi_\infty\mu_k$ is a weak order-unity in $\pi_\infty\Lambda$ for all $k \in \mathbb{N}$;*

(3) *there exists a partition $(\pi_k)_{k=0}^{\infty}$ of the projection $\pi_\infty^\perp$ such that $\pi_0\mu_1 = 0$, $\pi_k \le [\mu_k]$, and $\pi_k\mu_{k+1} = 0$, $k \in \mathbb{N}$;*

(4) *the representation is valid*

$$T = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} \mu_k e_k^\# \otimes f_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k e_k^\# \otimes f_k.$$

$\lhd$ By virtue of 3.4 we may assume that X and Y coincide with the bounded descents of Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. The operator $\mathcal{T} := T{\uparrow} \colon \mathcal{X} \to \mathcal{Y}$ is compact and we may apply inside $\mathbf{V}^{(B)}$ the ZFC-theorem on the general form of a compact operator in Hilbert space. Working inside $\mathbf{V}^{(B)}$ we may choose orthonormal sequences $(e_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$, $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{Y}$, and a decreasing numeric sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathcal{R}_+ \setminus 0$ such that $\lim \mu_k = 0$ and the presentation holds:

$$\mathcal{T} = \sum_{k=1}^{\infty} \mu_k e_k^* \otimes f_k.$$

Moreover, either $(\forall k \in \mathbb{N})\, \mu_k > 0$ or $(\exists k \in \mathbb{N})\, \mu_k = 0$. Since $[\![\, \mu_1 \le \|\mathcal{T}\| \,]\!] = \mathbf{1}$ we have $\mu_1 \le |T| \in \Lambda$, whence $(\mu_k) \subset \Lambda$. Let $\pi_\infty := [\![\, \mathcal{T}$ be an infinite-rank compact operator from a Hilbert space $\mathcal{X}$ to a Hilbert space $\mathcal{Y} \,]\!] = \mathbf{1}$. If $\mu'_k := \pi_\infty \mu_k$ then $[\![\, \mu'_k > 0 \,]\!] = [\![\, \mu'_k \ge \mu'_{k+1} \,]\!] = [\![\, \lim \mu'_k = 0 \,]\!] = \pi_\infty$, so that μ'_k is a weak order-unity in $\pi_\infty \Lambda$, $\mu'_k \ge \mu'_{k+1}$, and $o\text{-}\lim \mu'_k = 0$. From the above-indicated presentation for $\mathcal{T}$ we deduce

$$\pi_\infty T = bo\text{-}\sum_{k=1}^{\infty} \mu'_k e_k^{\#} \otimes f_k.$$

Consider the fragment $\pi_\infty^\perp T$. From the definition of π_∞ it follows that $\pi_\infty^\perp = [\![\, \mathcal{T}$ is a finite-rank operator $]\!] = \mathbf{1}$. The operator $\mathcal{T}$ has finite rank if and only if $\mu_n = 0$ for some $n \in \mathbb{N}$. Thus,

$$\pi_\infty^\perp = [\![\, (\exists n \in \mathbb{N}^\wedge)\, \mu_n = 0 \,]\!] = \bigvee_{n=1}^{\infty} [\![\, \mu_n = 0 \,]\!].$$

Put $\rho_n := [\![\, \mu_n = 0 \,]\!]$, $\pi_0 := \rho_1$, $\pi_n := \rho_{n+1} \Leftrightarrow \rho_n$, $(n \in \mathbb{N})$. Since $\pi_n = [\![\, \mu_{n+1} = 0 \,\&\, \mu_n \ne 0 \,]\!]$, we have construct a countable partition $(\pi_n)_{n=0}^\infty$ of the projection $\pi_\infty^\perp$ with $\pi_n \mu_{n+1} = 0$. Therefore, $\pi_n T = \sum_{k=1}^n \pi_n \mu_k e_k^{\#} \otimes f_k$ for all $n \in \mathbb{N}$. It remains to observe that $T = \pi_\infty T + bo\text{-}\sum_{n=0}^\infty \pi_n T$. $\rhd$

4. Fredholm B-alternative

A variant of the Fredholm Alternative holds for cyclically compact operators. We will call it the *Fredholm B-Alternative.*

4.1. Let X be a Banach space with the dual X^*. Take a bounded operator $T : X \to X$ and consider the equation of the first kind

$$Tx = y \quad (x, y \in X)$$

and the conjugate equation

$$T^*y^* = x^* \quad (x^*, y^* \in X^*).$$

The corresponding homogeneous equations are defined as $Tx = 0$ and $T^*y^* = 0$. Let $\varphi_0(T)$, $\varphi_1(n, T)$, $\varphi_2(n, T)$, and $\varphi_3(n, T)$ be set-theoretic formulas formalizing the following statements.

$\varphi_0(T)$: The homogeneous equation $Tx = 0$ has a sole solution, zero. The homogeneous conjugate equation $T^*y^* = 0$ has a sole solution, zero. The equation $Tx = y$ is solvable and has a unique solution given an arbitrary right side. The conjugate equation $T^*y^* = x^*$ is solvable and has a unique solution given an arbitrary right side.

$\varphi_1(n, T)$: The homogeneous equation $Tx = 0$ has n linearly independent solutions $x_1, \dots, x_n$. The homogeneous conjugate equation $T^*y^* = 0$ has n linearly independent solutions $y_1^*, \dots, y_n^*$.

$\varphi_2(n, T)$: The equation $Tx = y$ is solvable if and only if $y_1^*(y) = \dots = y_n^*(y) = 0$. The conjugate equation $T^*y^* = x^*$ is solvable if and only if $x^*(x_1) = \dots = x^*(x_n) = 0$.

$\varphi_3(n, T)$: The general solution x of the equation $Tx = y$ is the sum of a particular solution x_0 and the general solution of the homogeneous equation; i.e., it has the form

$$x = x_0 + \sum_{k=1}^{n} \lambda_k x_k \quad (\lambda_k \in \mathbb{C}).$$

The general solution y^* of the conjugate equation $T^*y^* = x^*$ is the sum of a particular solution y_0^* and the general solution of the homogeneous equation; i.e., it has the form

$$y^* = y_0^* + \sum_{k=1}^{n} \mu_k y_k^* \quad (\mu_k \in \mathbb{C}).$$

Using this notation, the Fredholm Alternative can be written as follows (see [4]):

$$\varphi_0(T) \vee (\exists n \in \mathbb{N})\, \varphi_1(n, T) \,\&\, \varphi_2(n, T) \,\&\, \varphi_3(n, T).$$

Thus, the conventional Fredholm Alternative distinguishes the cases $n = 0$ and $n \neq 0$. (If $n = 0$ then the formula

$$\varphi_1(n, T) \,\&\, \varphi_2(n, T) \,\&\, \varphi_3(n, T)$$

is equivalent to $\varphi_0(T)$.)

4.2. Consider now a B-cyclic Banach space X and a bounded B-linear operator T in X. In this case X and $X^{\#}$ are modules over the Stone algebra $\Lambda := \mathcal{S}(B)$ and

T is Λ-linear (= module homomorphism). A subset $\mathcal{E} \subset X$ is said to be *locally linearly independent* if whenever $e_1, \dots, e_n \in \mathcal{E}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, and $\pi \in B$ with $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ we have $\pi \lambda_k e_k = 0$ for all $k := 1, \dots, n$. We say that the *Fredholm B-Alternative* is valid for an operator T if there exists a countable partition of unity (b_n) in B such that the following conditions are fulfilled:

(1) The homogeneous equation $b_0 \circ Tx = 0$ has a sole solution, zero. The homogeneous conjugate equation $b_0 \circ T^\# y^\# = 0$ has a sole solution, zero. The equation $b_0 \circ Tx = b_0 y$ is solvable and has a unique solution given an arbitrary $y \in X$. The conjugate equation $b_0 \circ T^\# y^\# = b_0 x^\#$ is solvable and has a unique solution given an arbitrary $x^\# \in X^\#$.

(2) For every $n \in \mathbb{N}$ the homogeneous equation $b_n \circ Tx = 0$ has n locally linearly independent solutions $x_{1,n}, \dots, x_{n,n}$ and the homogeneous conjugate equation $b_n \circ T^\# y^\# = 0$ has n locally linearly independent solutions $y^\#_{1,n}, \dots, y^\#_{n,n}$ (hence have nonzero solutions).

(3) The equation $Tx = y$ is solvable if and only if $b_n \circ y^\#_{k,n}(y) = 0$ ($n \in \mathbb{N}$, $k \le n$). The conjugate equation $T^\# y^\# = x^\#$ is solvable if and only if $b_n \circ x^\#(x_{k,n}) = 0$ ($n \in \mathbb{N}$, $k \le n$).

(4) The general solution x of the equation $Tx = y$ has the form

$$x = bo\text{-}\sum_{n=1}^{\infty} b_n \left(x_n + \sum_{k=1}^{n} \lambda_{k,n} x_{k,n} \right),$$

where x_n is a particular solution of the equation $b_n \circ Tx = b_n y$ and $(\lambda_{k,n})_{n \in \mathbb{N}, k \le n}$ are arbitrary elements in Λ.

The general solution $y^\#$ of the conjugate equation $T^\# y^\# = x^\#$ has the form

$$y^\# = bo\text{-}\sum_{n=1}^{\infty} b_n \left(y^\#_n + \sum_{k=1}^{n} \lambda_{k,n} y^\#_{k,n} \right),$$

where $y^\#_n$ is a particular solution of the equation $b_n \circ T^\# y^\# = b_n x^\#$, and $\lambda_{k,n}$ are arbitrary elements Λ for $n \in \mathbb{N}$ and $k \le n$.

4.3. Theorem. *If S is a cyclically compact operator in a B-cyclic space X then the Fredholm B-Alternative is valid for the operator $T := I_X \Leftrightarrow S$.*

$\lhd$ Again we assume, without loss of generality, that X is the bounded part of the descent of a Banach space $\mathcal{X} \in \mathbf{V}^{(B)}$ and T is the restriction onto X of the descent of a bounded linear operator $\mathcal{T} \in \mathbf{V}^{(B)}$. Moreover, $[\![\mathcal{T} = I_{\mathcal{X}} \Leftrightarrow \mathcal{S}]\!] = \mathbf{1}$ and $[\![\mathcal{S}$ is a compact operator in $\mathcal{X}]\!] = \mathbf{1}$. We may assume that also $X = \mathcal{X}^* {\downarrow}^\infty$ and $T = \mathcal{T}^* {\downarrow}^\infty$, see [1; 5.5.10]. The Fredholm Alternative 4.1 is fulfilled for $\mathcal{T}$ inside

$\mathbf{V}^{(B)}$ by virtue of the Transfer Principle. In other words, the following relations hold:

$$\begin{aligned}\mathbf{1} &= [\![\varphi_0(\mathcal{T}) \vee (\exists n \in \mathbb{N}^{\wedge})\, \varphi_1(n,\mathcal{T}) \,\&\, \varphi_2(n,\mathcal{T}) \,\&\, \varphi_3(n,\mathcal{T})]\!] \\ &= [\![\varphi_0(\mathcal{T})]\!] \vee \bigvee_{n\in\mathbb{N}} [\![\varphi_1(n^{\wedge},\mathcal{T})]\!] \wedge [\![\varphi_2(n^{\wedge},\mathcal{T})]\!] \wedge [\![\varphi_3(n^{\wedge},\mathcal{T})]\!].\end{aligned}$$

Denote $b_0 := [\![\varphi_0(\mathcal{T})]\!]$ and $b_n := [\![\varphi_1(n^{\wedge},\mathcal{T})]\!] \wedge [\![\varphi_2(n^{\wedge},\mathcal{T})]\!] \wedge [\![\varphi_3(n^{\wedge},\mathcal{T})]\!]$. Since the formulas $\varphi_0(\mathcal{T})$ and $\varphi_1(n,\mathcal{T})$ & $\varphi_2(n,\mathcal{T})$ & $\varphi_3(n,\mathcal{T}))$ for different n are inconsistent, the sequence $(b_n)_{n=0}^{\infty}$ is a partition of unity in B. We will now prove that 4.2 (1–4) are valid.

(1): The claim 4.2 (1) is equivalent to the identities $\ker(T) = \{0\}$ and $\operatorname{im}(T) = X$ that are ensured by the following easy relations:

$$\mathbf{V}^{(B)} \models \ker(T){\uparrow} = \ker(\mathcal{T}) = \{0\}, \quad \mathbf{V}^{(B)} \models \operatorname{im}(T){\uparrow} = \operatorname{im}(\mathcal{T}) = \mathcal{X}.$$

(2): The part of the assertion $\varphi_1(n^{\wedge},\mathcal{T})$ concerning the solution of the equation $Tx = 0$ is formalized as

$$\begin{gathered}(\exists x)\,\big((x : \{1,\dots,n\}^{\wedge} \to \mathcal{X}) \,\&\, (\forall k \in \{1,\dots,n\}^{\wedge})\,(\mathcal{T}x(k) = 0) \\ \&\, \text{the set } x(\{1,\dots,n\}^{\wedge}) \text{ is linearly independent})\big).\end{gathered}$$

Moreover, there is no loss of generality in assuming that $\|x(k)\| \le 1$, $k \in \{1,\dots,n\}^{\wedge}$. Using the Maximum Principle and the properties of the modified descent we may find a mapping $\mathbf{x}$ from $\{1,\dots,n\}$ to X such that the image of the mapping $b_n\mathbf{x} : k \mapsto b_n\mathbf{x}(k)$ is a locally linearly independent set in X and $[\![\mathcal{T}\mathbf{x}(k) = 0]\!] \ge b_n$ for each $k \in \{1,\dots,n\}$. Put $x_{k,n} := b_n\mathbf{x}(k)$. Further,

$$[\![Tx_{k,n} = 0]\!] = [\![\mathcal{T}\mathbf{x}(k) = 0]\!] \wedge [\![\mathbf{x}(k) = x_{k,n}]\!] \ge b_n,$$

so that $b_nTx_{k,n} = 0$. The conjugate homogeneous equation is handled in the same fashion.

(3): Necessity of the stated conditions can be easily checked; prove sufficiency. We confine exposition to the equation $Tx = y$, since the conjugate equation is considered along similar lines. Suppose that $y^{\#}_{k,n}(y) = 0$ for $k,n \in \mathbb{N}$ and $k \le n$. Then

$$b_n \le [\![y^{\#}_{k,n}(y) = 0]\!] = [\![y^{\#}_{k,n}{\uparrow}\,(y) = 0]\!] \quad (k \in \{1,\dots,n\}).$$

At the same time, in view of (2), $[\![\{y^{\#}_{k,n} : k = 1,\dots,n\}{\uparrow}$ is a maximal linearly independent set of solutions of the equation $\mathcal{T}^*y^* = 0]\!] = \mathbf{1}$. All this implies that $[\![$ the equation $\mathcal{T}x = y$ is solvable $]\!] \ge b_n$, whence the equation $b_n \circ Tx = b_ny$ has at least one solution x_n. It is then easy to check that $\bar{x} := \sum_{n=1}^{\infty} b_n x_n$ is a solution of the equation $Tx = y$.

(4): If x is a solution of the equation $Tx = y$ then $[\![\, \mathcal{T}x = y \,]\!] = \mathbf{1}$. Taking into account the inequality $[\![\, \varphi_3(n^\wedge, \mathcal{T}) \,]\!] \geq b_n$, we arrive at

$$b_n \leq [\![(\exists\lambda)\,\big(\lambda : \{1,\dots,n\}^\wedge \to \mathcal{R}\,\&\, x = x^* + \sum_{k=1}^{n^\wedge} \lambda(k)u(k))]\!],$$

where u is the ascent of the mapping $k \mapsto x_{k,n}$ $(k = 1,\dots,n)$. The Maximum Principle guarantees the existence of a mapping ℓ_n from $\{1,\dots,n\}$ to Λ such that

$$[\![x = \bar{x} + \sum_{k=1}^{n^\wedge} \ell_n{\uparrow}\,(k)u(k)]\!] = \mathbf{1}.$$

Putting $\lambda_{k,n} := b_n \ell_n(k)$, we obtain

$$b_n x = b_n x_n + \sum_{k=1}^{n} \lambda_{k,n} b_n x_{k,n},$$

whence the desired representation follows. The general form of the solution of the conjugate equation is established by similar arguments. $\rhd$

5. Concluding remarks

5.1. The bounded descent of 1.4 appeared in the research by G. Takeuti into von Neumann algebras and C^*-algebras within Boolean-valued models [5, 6] and in the research by M. Ozawa into Boolean-valued interpretation of the theory of Hilbert spaces [7]. Theorem 3.4 on Boolean-valued representation of Kaplansky–Hilbert modules was proved by M. Ozawa [7].

5.2. Cyclically compact sets and operators in lattice-normed spaces were introduced in [8] and [3], respectively. Diffrernt aspects of cyclical compactness see in [9–12]. A standard proof of Theorem 2.4 can be extracted from [3] wherein more general approach is developed for the case of lattice normed space. Certain variants of Theorems 3.5 and 4.3 for operators in Banach–Kantorovich spaces can be also found in [3].

5.3. The famous result by P. G. Dodds and D. H. Fremlin [13] asserts that if a positive operator acting from a Banach lattice whose dual has order continuous norm to a Banach lattice with order continuous norm is dominated by a compact operator then the initial operator is also compact, see [14] for proof and related results. As regards cyclical compactness, we observe the conjecture of [15] that if a dominated operator T between spaces with mixed norm is cyclically compact and $|T| \leq S$ with S compact then T is also compact on assuming some conditions on the norm lattices like in the Dodds–Fremlin Theorem. This problem remains open.

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