Владикавказский математический журнал
апрель–июнь, 2007, Том 9, Выпуск 2

UDC 517.98

WHEN ALL SEPARATELY BAND PRESERVING BILINEAR OPERATORS ARE SYMMETRIC?

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To Vladimir Kojbaev on occasion of his 50th birthday

A purely algebraic characterization of universally complete vector lattices in which all separately band preserving bilinear operators are symmetric is obtained: this class consists of universally complete vector lattices with σ-distributive Boolean algebra of bands.

Mathematics Subject Classification (2000): 46A40, 47A65, 03C90, 06F25.

Key words: vector lattice, band preserving operator, orthosymmetric bilinear operator, σ-distributive Boolean algebra, Boolean valued model.

The aim of this note is to give an algebraic characterization of those universally complete vector lattice in which all band preserving bilinear operators are symmetric. We start with recalling some definitions and auxiliary facts about bilinear operators on vector lattices. For the theory of vector lattices and positive operators we refer to the books [1] and [7].

Let E and F be vector lattices. A bilinear operator $b : E\times E \to G$ is called *orthosymmetric* if $x \wedge y = 0$ implies $b(x, y) = 0$ for arbitrary $x, y \in E$, see [3]. Recall also that b is said to be *symmetric* (or *antisymmetric*) if $b(x, y) = b(y, x)$ (respectively $b(x, y) = -b(y, x)$) for all $x, y \in E$. Finally, b is said to be *positive* if $b(x, y) \geqslant 0$ for all $0 \leqslant x, y \in E$ and *orthoregular* if it can be represented as the difference of two positive orthosymmetric bilinear operators [2]. The vector space of all orthoregular bilinear operators and its subspaces are always considered with the ordering determined by the cone of positive operators.

The following important property of orthosymmetric bilinear operators is due to G. Buskes and A. van Rooij (see [3, Corollary 2]):

Proposition 1. *If E and F are arbitrary Archimedean vector lattices, then any positive orthosymmetric* (*and hence any orthoregular*) *bilinear operator from $E\times E$ to F is symmetric.*

A bilinear operator $b : E\times E \to E$ is said to be *separately band preserving* if the mappings $b(\cdot, e)$ and $b(e, \cdot)$ are band preserving for all $e \in E$ or, equivalently, if $b(L \times E) \subset L$ and $b(E \times L) \subset L$ for any band L in E. For linear band preserving operators see [1, 5, 7].

Proposition 2. *Let E be an Archimedean vector lattice and b is a bilinear operator in E* (*i. e. b acts from $E \times E$ into E*)*. Then the following assertions are equivalent:*

(1) *b is separately band preserving;*

(2) *$b(x, y) \in \{x\}^{\perp\perp} \cap \{y\}^{\perp\perp}$ for all $x, y \in E$;*

(3) *$b(x, y) \perp z$ for any $z \in E$ provided that $x \perp z$ or $y \perp z$;*

[1]The author is supported by a grant from Russian Foundation for Basic Research, project № 06-01-00622.

If E has the principal projection property, then (1)–(3) *are equivalent to* (4) *and* (5)*:*

(4) $\pi b(x,y) = b(\pi x, \pi y)$ *for any band projection π in E and all $x, y \in E$;*

(5) $\pi b(x,y) = b(\pi x, y) = b(x, \pi y)$ *for any band projection π in E and all $x, y \in E$.*

$\lhd$ We omit the routine arguments, cf. [1, Theorem 8.2]. $\rhd$

It was proved in [4, Theorem 4] that for each Archimedean vector lattice E there exists a unique (up to lattice isomorphism) square, i. e. a pair $(E^{\odot}, \odot)$, where $E^{\odot}$ is an Archimedean vector lattice and $\odot$ is a symmetric bimorphism from $E \times E$ to $E^{\odot}$, with the following universal property: if b is a symmetric lattice bimorphism from $E \times E$ to some Archimedean vector lattice F, then there is a unique lattice homomorphism $\Phi_b : E^{\odot} \to F$ with $b = \Phi_b \odot$. The bimorphism $\odot$ is an example of a separately band preserving bilinear operator, see [12, Theorem 6.4].

Proposition 3. *Let E be a relatively uniformly complete vector lattice with the square $E^{\odot}$. The correspondence $S \mapsto S\odot$ is an isomorphism of the vector lattice* $\mathrm{Orth}(E^{\odot})$ *onto the ordered vector space of all order bounded separately band preserving bilinear operators in E.*

$\lhd$ Follows from [12, Theorems 6.2 (2) and 6.4] and [4, Theorem 9]. $\rhd$

Evidently, a separately band preserving bilinear operator is orthosymmetric. Hence, all orthoregular separately band preserving operators are symmetric by Proposition 1. This brings up the question, which can be considered as a version of *Wickstead's problem* (see [7, 10, 11]):

Problem. *Under what conditions all separately band preserving bilinear operators in a vector lattice are symmetric? order bounded?*

A Boolean σ-algebra $\mathbb{B}$ is called *σ-distributive* if

$$\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b_{n,m} = \bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \bigvee_{n\in\mathbb{N}} b_{n,\varphi(n)}$$

for any double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{B}$. Other equivalent definitions are collected in [15], see also [7]. Now, we are able to state the main result of the note, cf. [10, 11].

Theorem. *Let G be a universally complete vector lattice and $\mathbb{B} := \mathfrak{B}(G)$ denotes the complete Boolean algebra of all bands in G. Then the following are equivalent:*

(1) $\mathbb{B}$ *is σ-distributive;*

(2) *there is no nonzero separately band preserving antisymmetric bilinear operator in G;*

(3) *all separately band preserving bilinear operators in G are symmetric;*

(4) *all separately band preserving bilinear operators in G are order bounded.*

Our proof of the stated theorem uses the Boolean valued approach which consists primarily in comparison of the instances of a mathematical object in two different Boolean valued models, most commonly the classical *von Neumann universe* $\mathbb{V}$ and the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$. All necessary information from Boolean values analysis can be found in [13].

Fix a complete Boolean algebra $\mathbb{B}$ and consider the corresponding Boolean valued model of set theory $\mathbb{V}^{(\mathbb{B})}$. Let $\mathscr{R}$ be the field of reals inside $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order, see [13]), is a universally complete vector lattice. Recall that $X \mapsto X^{\wedge}$ denotes the *standard name mapping* which embeds $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$. It is well known that if $\mathbb{R}$ is the field of reals in $\mathbb{V}$, then $\mathbb{R}^{\wedge}$ can be considered as a dense subfield of $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$.

Lemma 1. *A Boolean algebra $\mathbb{B}$ is σ-distributive if and only if* $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^{\wedge}$.

$\lhd$ This fact was obtained by A. E. Gutman [5, 6], see also [7]. $\rhd$

Let $BL_N(G)$ stands for the set of all separately band preserving bilinear operators in $G := \mathscr{R}{\downarrow}$. Clearly, $BL_N(G)$ becomes a faithful unitary module over the ring G if we define gT

as $gT : x \mapsto g \cdot Tx$ for all $x \in G$. Denote by $BL_{\mathbb{R}^\wedge}(\mathscr{R})$ the element of $\mathbb{V}^{(\mathbb{B})}$ representing the space of all internal $\mathbb{R}^\wedge$-bilinear mappings from $\mathscr{R} \times \mathscr{R}$ to $\mathscr{R}$. Then $BL_{\mathbb{R}^\wedge}(\mathscr{R})$ is a vector space over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$, and $BL_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$ is an external (= in $\mathbb{V}$) faithful unitary module over G.

Lemma 2. *The modules $BL_N(G)$ and $BL_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$ are isomorphic by sending each separately band preserving bilinear operator b to its ascent $b{\uparrow}$.*

$\lhd$ We make use of the same arguments as in [8, Proposition 3.3]. Denote by $[e]$ the order projection onto the band $\{e\}^{\perp\perp}$. Proposition 2 implies that every $b \in BL_N(G)$ is extensional:

$$[b(x,y) - b(u,v)] \leqslant [x-u] \vee [y-v] \quad (x,y,u,v \in G).$$

Therefore, (since every extensional mapping has the ascent) there exists a unique internal function $\beta := b{\uparrow}$ from $\mathscr{R} \times \mathscr{R}$ to $\mathscr{R}$ such that $\mathbb{V}^{(\mathbb{B})} \models \beta(x,y) = b(x,y)$ $(x,y \in G)$. With this in mind we deduce ($\oplus$ and $\odot$ stand for internal operations in $\mathscr{R}$):

$$\beta(x \oplus y, z) = b(x+y,z) = b(x,z) + b(y,z) = \beta(x,z) \oplus \beta(y,z) \quad (x,y,z \in G)$$
$$\beta(\lambda^\wedge \odot x, z) = b(\lambda \cdot x, z) = \lambda \cdot b(x,z) = \lambda^\wedge \odot \beta(x,z) \quad (x,z \in G, \lambda \in \mathbb{R}).$$

Thus, $[\![\, \beta : \mathscr{R} \to \mathscr{R}$ is a $\mathbb{R}^\wedge$-bilinear function$\,]\!] = \mathbb{1}$, i. e. $[\![\beta \in BL_{\mathbb{R}^\wedge}(\mathscr{R})]\!] = \mathbb{1}$. Conversely, if $\beta \in BL_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$, then the descent $\beta{\downarrow} : G \times G \to G$ is extensional, since the descent of any mapping is extensional, and bilinear, since β is $\mathbb{R}^\wedge$-bilinear inside $\mathbb{V}^{(\mathbb{B})}$. Moreover, we have

$$(\forall\, s,t \in \mathscr{R})\, st = 0 \ \to\ \beta(s,t) = 0.$$

Interpreting this in $\mathbb{V}^{(\mathbb{B})}$ we obtain that $b = \beta{\downarrow}$ is orthosymmetric. Now it remains to observe that an orthosymmetric extensional bilinear operator is separately order preserving. $\rhd$

Lemma 3. *Let $\mathbb{P}$ be a subfield of $\mathbb{R}$ and let $\mathscr{E}$ be a Hamel basis of the vector space $\mathbb{R}$ over the field $\mathbb{P}$. The general form of a $\mathbb{P}$-bilinear function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by*

$$\beta(x,y) = \sum\nolimits_{e_1,e_2 \in \mathscr{E}} x_{e_1} y_{e_2} \phi(e_1,e_2), \quad x = \sum\nolimits_{e \in \mathscr{E}} x_e e,\ y = \sum\nolimits_{e \in \mathscr{E}} y_e e,$$

where $\phi : \mathscr{E} \times \mathscr{E} \to \mathbb{P}$ is an arbitrary function with finite number of nonzero values.

$\lhd$ Follows easily from the definition of bilinear operator and the properties Hamel basis. $\rhd$

$\lhd$ Proof of the Theorem. Suppose that an order unit $\mathbb{1}$ is fixed in G and G is endowed with the multiplication that makes G an f-algebra having $\mathbb{1}$ as its unit element.

(1) $\to$ (4): Let $b : G \times G \to G$ be a band preserving bilinear operator and put $c := b(\mathbb{1},\mathbb{1})$. We may assume that G is locally one-dimensional, since this property is equivalent to the assertion (1) as was shown by A. E. Gutman [5]. In that event for arbitrary $x,y \in G$ there exists a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathfrak{P}(G)$ and two families of reals $(s_\xi)_{\xi \in \Xi}$ and $(t_\xi)_{\xi \in \Xi}$ such that $\pi_\xi x = s_\xi \pi_\xi \mathbb{1}$ and $\pi_\xi y = t_\xi \pi_\xi \mathbb{1}$ for all $\xi \in \Xi$. Proposition 2 implies that $\pi_\xi b(x,y) = s_\xi t_\xi \pi_\xi c$ for all $\xi \in \Xi$ and hence $b(x,y) = cxy$. Now it evident that b is order bounded.

(4) $\to$ (3): A separately band preserving bilinear operator in G is order bounded by (4) and hence orthosymmetric; therefore, it is symmetric by Proposition 1.

(3) $\to$ (2): Any separately band preserving bilinear operator is symmetric by (3) and hence it is equal to zero, provided that it is also antisymmetric.

(2) $\to$ (1): Assume that $\mathbb{B}$ is not σ-distributive. Then $\mathbb{R}^\wedge \neq \mathscr{R}$ by Lemma 1 and a separately band preserving antisymmetric bilinear operator can be constructed on using Lemma 3. Indeed, a Hamel bases $\mathscr{E}$ of $\mathscr{R}$ over $\mathbb{R}^\wedge$ contains at least two different elements $e_1 \neq e_2$. Define a function $\phi : \mathscr{E} \times \mathscr{E} \to \mathscr{R}$ so that $1 = \phi(e_1,e_2) = -\phi(e_2,e_1)$, and $\phi(e_1',e_2') = 0$ for all

other pairs $(e'_1, e'_2) \in \mathscr{E} \times \mathscr{E}$ (in particular, $0 = \phi(e_1, e_1) = \phi(e_2, e_2)$). By Lemma 3 β_0 can be extended to an $\mathbb{R}^\wedge$-bilinear function $\beta : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$. The descent b of β is a separately band preserving bilinear operator in G by Lemma 2. Moreover, b is nonzero and antisymmetric, since β is nonzero and antisymmetric by construction. This contradiction proves that $\mathbb{R}^\wedge = \mathscr{R}$ and $\mathbb{B}$ is σ-distributive. $\rhd$

Corollary 1. *There exists a nonatomic universally complete vector lattice in which all separately band preserving bilinear operators are order bounded and hence symmetric.*

$\lhd$ It follows from the above Theorem and the following result by A. E. Gutman [5, 6]: there exists a nonatomic locally one-dimensional universally complete vector lattice. $\rhd$

A bilinear operator $b : G \times G \to G$ is called *essentially nontrivial* if $\pi b = 0$ implies $\pi = 0$ for any band projection $\pi \in \mathfrak{P}(G)$. The definition of a locally separable measure space see in [9].

Corollary 2. *Let* (Ω, Σ, μ) *be a nonatomic locally separable measure space and let* $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$ *be the vector space of all equivalence classes of* (*almost everywhere equal*) *real measurable functions. Then there exists an essentially nontrivial separately band preserving antisymmetric bilinear operator in* $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$.

$\lhd$ The proof goes in much the same way as in [9]. $\rhd$

References

1. *Aliprantis C. D., Burkinshaw O.* Positive Operators.—New York: Acad. Press, 1985.
2. *Buskes G., Kusraev A. G.* Representation and extension of orthoregular bilinear operators // Vladikavkaz Math. J.—2007.—V. 9, № 1.—P. 16–29.
3. *Buskes G., van Rooij, A.* Almost f-algebras: commutativity and the Cauchy–Schwarz inequality // Positivity.—2000.—V. 4.—P. 227–231.
4. *Buskes G., van Rooij A.* Squares of Riesz spaces // Rocky Mountain J. Math.—2001.—V. 31, № 1.—P. 45–56.
5. *Gutman A. E.* Locally one-dimensional K-spaces and σ-distributive Boolean algebras // Siberian Adv. Math.—1995.—V. 5, № 2.—P. 99–121.
6. *Gutman A. G.* Disjointness preserving operators // Vector Lattices and Integral Operators (ed. S. S. Kutateladze).—Dordrecht etc.: Kluwer, 1996.—P. 361–454.
7. *Kusraev A. G.* Dominated Operators.—Dordrecht: Kluwer, 2000.
8. *Kusraev A. G.* On band preserving operators // Vladikavkaz Math. J.—2004.—V. 6, № 3.—P. 47–58.
9. *Kusraev A. G.* Automorphisms and derivations in the algebra of complex measurable functions // Vladikavkaz Math. J.—2005.—V. 7, № 3.—P. 45–49.
10. *Kusraev A. G.* Automorphisms and derivations in universally complete complex f-algebras // Siberian Math. J.—2006.—V. 47, № 1.—P. 97–107.
11. *Kusraev A. G.* Analysis, algebra, and logics in operator theory // Complex Analysis, Operator Theory, and Mathematical Modeling (Eds. Yu. F. Korobeĭnik, A. G. Kusraev).—Vladikavkaz: Vladikavkaz Scientific Center, 2006.—P. 171–204.
12. *Kusraev A. G.* Orthosymmetric Bilinear Operators.—Vladikavkaz: IAMI VSC RAS, 2007. (Preprint).
13. *Kusraev A. G., Kutateladze S. S.* Boolean valued analysis.—Dordrecht: Kluwer, 1995.
14. *Kusraev A. G., Tabuev S. N.* On disjointness preserving bilinear operators // Vladikavkaz Math. J.—2004.—V. 6, № 1.—P. 58–70.
15. *Sikorski R.* Boolean Algebras.—Berlin etc.: Springer-Verlag, 1964.
16. *Wickstead A. W.* Representation and duality of multiplication operators on Archimedean Riesz spaces // Compositio Math.—1977.—V. 35, №. 3.—P. 225–238.

Received 22 March 2007

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