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ON RIESZ SPACES WITH b-PROPERTY AND b-WEAKLY COMPACT OPERATORS

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An operator $T : E \to X$ between a Banach lattice E and a Banach space X is called b-weakly compact if $T(B)$ is relatively weakly compact for each b-bounded set B in E. We characterize b-weakly compact operators among o-weakly compact operators. We show summing operators are b-weakly compact and discuss relation between Dunford–Pettis and b-weakly compact operators. We give necessary conditions for b-weakly compact operators to be compact and give characterizations of KB-spaces in terms of b-weakly compact operators defined on them.

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Introduction

Riesz spaces considered in this note are assumed to have separating order duals. The order dual of a Riesz space E is denoted by $E^\sim$. $E^{\sim\sim}$ will denote the order bidual of E. The order continuous dual of E is denoted by $E_n^\sim$, while E' will denote the topological dual of a topological Riesz space. E_+ will denote the cone of positive elements of E. The letters E, F will denote Banach lattices, X, Y will denote Banach spaces. B_X will denote the closed unit ball of X. We use without further explanation the basic terminology and results from the theory of Riesz spaces as set out in [1, 2, 14, 17].

Let E be a Riesz subspace of a Riesz space F. A subset of E which is order bounded in F is said to be b-bounded in E. If every b-bounded subset of E remains to be order bounded in E then E is said to have b-property in F. If a Riesz space E has b-property in its order bidual $E_n^\sim$ then it is said to have b-property.

Riesz spaces with b-property were introduced in [3] and studied in [3–6].

A normed Riesz space E has the weak Fatou property for directed sets if every norm bounded upwards directed set of positive elements in E has a supremum. Riesz spaces with weak Fatou Property for directed sets have b-property. If a Banach lattice has order continuous norm then it has the weak Fatou property for directed sets if and only if it has the b-property [6]. A locally solid Riesz space is said to have Levi property if every topologically bounded set in E_+ has a supremum. If E is a Frechet lattice with Levi property then E has the b-property [6]. If E is a Dedekind complete locally solid Riesz space with $E' = E^\sim$ then E has b-property if and only if E has the Levi property [6]. Thus a Dedekind complete Frechet lattice has Levi property if and only if it has the b-property.

Let E be a Riesz subspace of a Riesz space F. If E is the range of a positive projection defined on F then E has b-property in F. If E is a Banach lattice then every sublattice of E isomorphic to l_1 has b-property in E [14, Proposition 2.3.11]. Similarly if the norm of E is order continuous then every sublattice Riesz isomorphic to c_0 has b-property in E [14, Proposition 2.4.3].

Further examples of Riesz spaces with b-property are given in the following example.

Example. A Banach lattice E is called a *KB*-space if every increasing norm bounded sequence in E_+ is norm convergent. *KB*-spaces have b-property. Perfect Riesz spaces have b-property and hence, every order dual has b-property [4]. If K is a compact Hausdorff space and $C(K)$ is the Riesz space of real valued continuous functions on K under pointwise order and algebraic operations then $C(K)$ has b-property[4]. On the other hand c_0 real sequences which converge to zero does not have b-property.

An element $e > 0$ in a Riesz space E is called discrete if the ideal generated by e coincides with the subspace generated by e. A Riesz space E is called discrete if and only if there exists a discrete element v with $0 < v < e$ for every $0 < e$ in E.

Example. Discrete elements give rise to ideals with b-property in a Riesz space E. Because if x is a discrete element then the principal ideal I_x generated by x is projection band in E and therefore I_x has b-property in E.

$T : E \to F$ is called b-bounded if T maps b-order bounded subsets of E into b-bounded subsets of F.

$T : E \to X$ is called b-weakly compact if T maps b-order bounded subsets of E into relatively weakly compact subsets of X.

Although the authors were not aware of this fact until quite recently, much later then the Bolu meeting in fact, b-weakly compact operators were introduced in [15] for the first time under a different name.These operators were studied in [4–11] and in [13–15]. Among b-weakly compact operators $T : E \to X$ those that map the band B generated by E in E'' into X are called strong type B in [15]. To describe the operators of strong type B, we refer the reader to [13].

A continuous operator $T : E \to X$ is called order weakly (o-weakly) compact whenever $T[0, x]$ is a relatively weakly compact subset of X for each $x \in E_+$.

A continuous operator $T : E \to X$ is called *AM*-compact if $T[-x, x]$ is relatively norm compact in X for each $x \in E_+$.

A continuous operator T from a Banach lattice E into a Banach lattice F is called semicompact if for every $\epsilon > 0$, there exists some $u \in E_+$ such that $T(B_E) \subseteq [-u, u] + \epsilon B_F$.

A continuous operator $T : X \to Y$ is called a Dunford-Pettis operator if $x_n \to 0$ in $\sigma(X, X')$ implies $\lim_n \|T(x_n)\| = 0$.

A b-weakly compact operator is continuous and if $W(E, X)$ is the space of weakly compact, $W_b(E, X)$ is the space of b-weakly compact and $W_0(E, X)$ is the space of order weakly compact operators we have the following relations between these classes of operators:

$$W(E, X) \subseteq W_b(E, X) \subseteq W_0(E, X).$$

The inclusions may be proper. The identity on $L^1[0, 1]$ is b-weakly compact but not weakly compact. The identity on c_0 is o-weakly compact but not a b-weakly operator.

If E is an *AM*-space then $W(E, X) = W_b(E, X)$. On the other hand Theorem 2.2. in [10] shows that if E' is a *KB*-space or X is reflexive then $W(E, X) = W_b(E, X)$. A Banach lattice E is a *KB*-space if and only if $L(E, X) = W_b(E, X)$ for each Banach space X [5]. If F is a *KB*-space then again $L(E, F) = W_b(E, F)$ for each Banach lattice E [5]. To generalize, we know that if a Banach space X does not contain c_0, then $L(E, X) = W_b(E, X)$.

We need the following characterization of b-weakly compact operators which is a combination of results in [3, 5].

Proposition 1. *Let $T : E \to X$ be an operator. The following are equivalent:*

1) *T is b-weakly compact.*

2) *For each b-bounded disjoint sequence (x_n) in E_+, $\lim_n q_T(x) = 0$ where $q_T(x)$ is the Riesz seminorm defined as $\sup\{\|T(y)\| : |y| \leqslant |x|\}$ for each $x \in E$.*

3) *$T(x_n)$ is norm convergent for each b-bounded increasing sequence (x_n) in E_+.*

4) *For each b-bounded disjoint sequence (x_n) in E, we have* $\lim_n \|T(x_n)\| = 0$.

b-weakly compact operators satisfy the domination property. That is, if $0 \leqslant S \leqslant T$ and T is b-weakly compact then S is also b-weakly compact which can be seen from the characterization given in Proposition 1(4).

Main results

A Riesz space E is called σ-laterally complete if the supremum of every disjoint sequence of E_+ exists in E. A Riesz space that is both σ-laterally and σ-Dedekind complete is called σ-universally complete. There exists a universally complete Riesz space E^u which contains E as an order dense Riesz subspace. E^u is called the universal completion of E.

The next result exhibits the relation between b-property and σ-lateral completeness. It is actually Theorem 23.23 in [1]. Restated for our purposes it reads as follows.

Proposition 2. *Let E be a σ-Dedekind complete Riesz space. Then E is σ-laterally complete if and only if E has b-property in its universal completion E^u.*

The following is Theorem 23.24 in [1].

Corollary. *Let E be a Dedekind complete Riesz space. Then E is universally complete if and only if E has countable b-property in E^u and has a weak order unit.*

$\lhd$ As E is order dense in the universal completion E^u, E is an order ideal of E^u by Theorem 2.2 in [1]. Suppose E has b-property in E^u and has a weak order unit e. Let $0 < u' \in E^u$ be that arbitrary. As e is also a weak order unit of E^u, we have $0 \leqslant u' \wedge ne \uparrow u'$. Since E is an ideal, $\{u' \wedge ne\} \subseteq E$ and since E has b-property in E^u, $\{u' \wedge ne\}$ is an order bounded subset of E and therefore $u' \in E$. Hence $E = E^u$. $\rhd$

Examples in [1] show that Dedekind completeness of E and existence of a weak order unit can not be omitted. Theorem 23.32 in [1] shows that among σ-laterally complete Riesz spaces those admitting a Riesz norm or an order unit are those which are Riesz isomorphic to $\mathbb{R}^n$. Thus if E is σ-Dedekind complete and has countable b-property in E^u which either has an order unit or admits a Riesz norm then E is isomorphic to $\mathbb{R}^n$.

Each order weakly compact operator $T : E \to X$ factors over a Banach lattice F with order continuous norm as $T = SQ$ where Q is an almost interval preserving lattice homomorphism which is the quotient map $E \to E/q_T^{-1}(0)$ in fact, F is the completion of $E/q_T^{-1}(0)$, where $q_T(x)$ is the Riesz seminorm defined as $\sup\{\|T(y)\| : |y| \leqslant |x|\}$ for each $x \in E$ and S is the operator mapping the equivalence class $[x]$ in $E/q_T^{-1}(0)$ to $T(x)$ [14,Theorem 3.4.6]. As b-weakly compact operators are order weakly compact every b-weakly compact operator $T : E \to X$ has a factorization $T = SQ$ over a Banach lattice with order continuous norm. Let us note that if E has order continuous norm then the factorization can be made over a *KB*-space as if was shown in [7].

This factorization yields a characterization of b-weakly compact operators among order weakly compact operators.

Proposition 3. *Let $T : E \to F$. T is b-weakly compact if and only if the quotient map $Q : E \to F$ is b-weakly compact.*

$\lhd$ Let F be the completion of $F_0 = E/q_T^{-1}(0)$ and Q be the quotient map $Q : E \to F_0$. Since Q is onto, the corresponding operator $Q : E \to F$ is an almost interval preserving lattice homomorphism.

Suppose T is b-weakly compact and let $(x_n) \subseteq E_+$ be an b-order bounded disjoint sequence. In view of $\|Q(x_n)\| = q_T(x_n)$, we see that $\lim_n \|Q(x_n)\| = 0$. Thus Q is b-weakly compact by Proposition 1(4).

On the other hand if Q is b-weakly compact then it is easily seen that SQ is also b-weakly compact for each continuous operator S, and thus $T = SQ$ is b-weakly compact. $\rhd$

This leads us to recapture a result of [5].

Corollary. *Suppose that $T : E \to F$ is b-weakly compact where F is a Dedekind complete AM-space with order unit.Then $|T|$ is a b-weakly compact operator.*

$\lhd$ T has a factorization over a Banach lattice H with order continuous norm as SQ where $Q : E \to H$ is b-weakly compact and $S : H \to F$ is continuous. Thus $|S|$ exists. The operator $|S|Q$ is b-weakly compact and $0 \leqslant |T| = |SQ| \leqslant |S|Q$. Thus $|T|$ is a b-weakly compact as b-weakly compact operators satisfy the domination property. $\rhd$

A deficiency of b-weakly compact operators is that they do not satisfy the duality property. For example, the identity I on l_1 is b-weakly compact but its adjoint, the identity on l_∞, is not b-weakly compact. On the other hand the identity on c_0 is not b-weakly compact but its adjoint, the identity on l_1, is certainly b-weakly compact. For recent developments on duality of b-weakly compact operators we refer the reader to [9].

One of the sufficients conditions for an operator to be b-weakly compact is that for each b-bounded disjoint sequence (x_n) in the domain we have $\lim_n \|T(x_n)\| = 0$. Utilizing this it is easy to see that b-weakly compact operators are norm closed in $L(E, X)$. A result in [12] shows that strong limit of o-weakly compact operators is also o-weakly compact under certain conditions. The following example shows that b-weakly compact operators behave differently in this respect.

Example.For each n, let $T_n : c_0 \to c_0$ be defined as $T_n(y) = (y_1, \ldots, y_n, 0, \ldots)$. Then the finite rank operators (T_n) are b-weakly compact for each n and we have $T_n(y) \to I(y)$ for each y in c_0. However the identity operator I on c_0 is not a b-weakly compact operator.

We will call an operator $T : E \to X$ summing if T maps weakly summable sequences in E to summable sequences in X.

Proposition 4. *Let $T : E \to X$ be a summing operator between a Banach lattice E and a Banach space X. Then T is b-weakly compact.*

$\lhd$ Let (e_n) be a b-bounded disjoint sequence in E_+. It suffices to show that $(T(x_n))$ is norm convergent to 0. There exists an e in E''_+ such that $0 \leqslant \sum e_k \leqslant e$ for each partial sum. It follows that the sequence (e_k) is a weakly summable sequence in E. As T is summing, we have $\sum Te_k < \infty$, and hence $\|Te_k\| \to 0$ in X. $\rhd$

It is easy to see that an operator $T : E \to X$ is b-weakly compact if and only if the operator $j_X T : E \to X''$ is b-weakly compact where j_X is the canonical embedding of X into X''. Let us recall that an operator $T : E \to X$ is called injective if T is one-to-one and has closed range. Generalizing the previous observation slightly we show that for an operator to be b-weakly compact the size of the target space does not matter.

Proposition 5. *Let $T : E \to X$ and $j : X \to Y$ be operators where j is an injection. Then T is b-weakly compact if and only if jT is b-weakly compact.*

Using the characterization of b-weakly compact operators given in Proposition 1(4) it follows immediately that every Dunford-Pettis operator $T : E \to X$ is actually a b-weakly compact operator. On the other hand the result in [11] shows that if E has weakly sequentially continuous lattice operations and has an order unit then every positive order weakly compact, in particular every b-weakly compact operator $T : E \to X$ is a Dunford-Pettis operator. Let us note however that weak sequential continuity of the lattice operations only is not sufficient. Indeed, the identity operator on c_0 is o-weakly compact but not a Dunford-Pettis operator although c_0 has weakly continuous lattice operations.

In opposite direction we have the following result which is a slight improvement of theorem 2.1 in [11].

Proposition 6. *If each positive b-weakly compact operator $T : E \to F$ is a Dunford-Pettis operator then either E has weakly sequentially continuous lattice operations or F has order continuous norm.*

$\lhd$ Let S and T be two operators from E into F satisfying $0 \leqslant S \leqslant T$ and T be a Dunford-Pettis operator.Then T is a b-weakly compact operator. As b-weakly compact operators satisfy the domination property S is also a b-weakly compact operator. By the assumption S is a Dunford-Pettis operator. The result now follows from Theorem 3.1 in [16]. $\rhd$

Now we investigate the relation between b-weakly compact operators and AM-compact operators. The natural embedding $j : L^\infty[0,1] \to L^p[0,1], 1 \leqslant p < \infty$ is a b-weakly compact operator which is not AM-compact.

Proposition 7. *Let E, F be Banach lattices with E' discrete. Then every o-weakly compact (and therefore every b-weakly compact) operator from E into F is AM-compact.*

$\lhd$ It suffices to show that $T[0,x]$ is relatively norm compact for each $x \in E_+$. Let S be the restriction of T to the principal order ideal I_x generated by x. Then $S : I_x \to F$ and $S' : F' \to I'_x$ are both weakly compact operators. Therefore $S'(B_{F'})$ is relatively compact in $\sigma(I_x, I''_x)$. I'_x is an AL-space. Let A be the solid hull of $S'(B_{F'})$ in I'_x. Every disjoint sequence in A is convergent for the norm in I'_x by Theorem 21.10 in [1]. Since E' is assumed to be discrete, A is contained in the band generated by discrete elements of I'_x. Employing Theorem 21.15 in [1], we see that A is relatively compact for the norm of I'_x. Therefore $S' : F' \to I'_x$ is a compact operator. Consequently, $T : I_x \to F$ is also compact and thus $T[0,x]$ is relatively compact in F. $\rhd$

If $T : E \to E$ is a b-weakly compact operator then T^2 is also a b-weakly compact but not necessarily a weakly compact operator. For example the identity I on $L^1[0,1]$ is b-weakly compact as $L^1[0,1]$ is a *KB*-space [3], but I^2 is not a weakly compact operator. It has recently been shown that for a positive b-weakly compact operator $T : E \to E$, T^2 is weakly compact if and only if each positive b-weakly compact operator $T : E \to E$ is weakly compact [10, Theorem 2.8].

Now we will now study compactness of b-weakly compact operators.

Proposition 8. *Suppose that every positive b-weakly compact operator is compact. Then one of the following holds:*

1) *E' and F have order continuous norms.*
2) *E' is discrete and has order continuous norm.*
3) *F is discrete and has order continuous norm.*

$\lhd$ Let $S, T : E \to F$ be such that $0 \leqslant S \leqslant T$ where T is compact. Then T and S are b-weakly compact operators. Thus S is compact by the hypothesis. The conclusion now follows from Theorem 2.1 in [16]. $\rhd$

On the compactness of squares of b-weakly compact operators we have the following. The proof is very similar to the proof of the preceding proposition. Therefore it is omitted.

Proposition 9 . *Let E be a Banach lattice with the property that for each positive b-weakly compact operator $S : E \to E$, S^2 is compact. Then one of the following holds.*

1) *E has order continuous norm.*

2) *E' has order continuous norm.*

3) *E' is discrete.*

b-property has been very useful in characterizing *KB*-spaces. For example a Banach lattice E is a *KB*-space if and only if E has order continuous norm and b-property or if and only if the identity operator on E is b-weakly compact [3–4].

We now present another characterization of *KB*-spaces.

Proposition 10. *A Banach lattice F is a KB-space if and only if for each Banach lattice E and positive disjointness preserving operator $T : E \to F$, T is b-weakly compact.*

$\lhd$ If the hypothesis on F is true then taking $E = F$, we see that the identity on E is b-weakly compact and thus E is a *KB*-space [3]. On the other hand if (x_n) is a b-bounded disjoint sequence in E_+, then (Tx_n) is an order bounded disjoint sequence in F as there exists a positive projection of F'' onto F. Then $\|T(x_n)\| \to 0$ as a *KB*-space has order continuous norm. It follows from Proposition 1(4) that T is b-weakly compact. $\rhd$

Proposition 11. *Consider operators $T : E \to F$ and $S : F \to G$. Suppose S is strong type B and T' is b-weakly compact. Then ST is a weakly compact operator.*

$\lhd$ It suffices to show $(ST)''(E'') \subseteq G$. Since order dual of a Banach lattice has b-property, T' is o-weakly compact and being so, T has factorization over a Banach lattice H with order continuous dual norm as $T = T_1T_0$ where $T_0 : E \to H$ is continuous and $T_1 : H \to F$ is an interval preserving lattice homomorphism by Theorem 3.5.6 in [14]. Since H' has order continuous norm, we have $(H')'_n = H''$ and $T_1''((H')'_n) \subseteq (F')'_n$ as T_1'' is order continuous. Now the weak compactness of ST follows from

$$(ST)''(E'') = S''(T_1''(T_0''(E''))) \subseteq S''(T_1''(H'')) \subseteq S''(T_1''(H')'_n) \subseteq S''(F')'_n \subseteq G$$

where the last inclusion follows from the fact that S is of strong type B and therefore S'' maps the band $(F')'_n$ generated by F in F'' into G. $\rhd$

As order duals have b-property, assuming T' to be b-weakly compact is the same as assuming it to be o-weakly compact. Also, we could have taken T to be semicompact as T' is o-weakly compact whenever T is semicompact [14, Theorem 3.6.18].

Corollary. *Let T be an operator on a Banach lattice such that both T and T' are strong type B. Then T^2 is weakly compact.*

Finally we study the relationship between semicompact and b-weakly compact operators. It is immediate from the definitions and Theorem 14.17 in [2] that if the range has order continuous norm, thus ensuring weak compactness of order intervals, each semicompact operator is weakly and therefore b-weakly compact.

On the other hand the identity I on l_1 is a b-weakly compact operator which is not semicompact. Theorem 127.4 in [17] shows that if E' and F have order continuous norms then every order bounded semicompact operator $T : E \to F$ is b-weakly compact.

The next result gives necessary and sufficient conditions for a Banach lattice to be a *KB*-space as well as illuminates the relation between semicompact and b-weakly compact operators.

First we need a Lemma which was first proved in [9].

Lemma. *Let E be a Banach lattice. If (e_n) is a positive disjoint sequence in E such that $\|e_n\| = 1$ for all n, then there exists a positive disjoint sequence (g_n) in E' with $\|g_n\| \leqslant 1$ and satisfying $g_n(e_n) = 1$ and $g_n(e_m) = 0$ for all $n \neq m$.*

$\lhd$ Let (e_n) be a disjoint sequence in E_+ with $\|e_n\| = 1$ for all n. By Hahn-Banach Theorem there exists $f_n \in E'_+$ such that $\|f_n\| = 1$ and $f_n(e_n) = \|e_n\| = 1$. Considering E in $(E')'_n$, we see that carriers C_{e_n} of e_n are mutually disjoint bands in E'. If g_n is the projection of f_n onto C_{e_n}, then the sequence (g_n) has the desired properties. $\rhd$

Let us recall that a Banach lattice E is said to have the Levi Property if every increasing norm bounded net in E_+ has a supremum in E_+. It is well-known that a Banach lattice with Levi Property is Dedekind complete.

The following result gives a necessary and sufficient conditions for a Banach lattice to be a *KB*-space.

Proposition 12. *Let E and F be Banach lattices and assume that F has the Levi property. Then the following are equivalent:*

1) *Each continuous operator $T : E \to F$ is b-weakly compact.*
2) *Each continuous semicompact operator $T : E \to F$ is b-weakly compact.*
3) *Each positive semicompact operator $T : E \to F$ is b-weakly compact.*
4) *Either E or F is a KB-space.*

$\lhd$ It is clear that 1) implies 2) and 2) implies 3). The implication 4) $\Rightarrow$ 2) was proved in [5]. We will prove that 3) implies 4).

Let us assume that neither E nor F is a *KB*-space. To finish the proof we construct a positive semicompact operator $T : E \to F$ which is not b-weakly compact. Recall that a Banach lattice is a *KB*-space if and only if the identity operator on it is b-weakly compact[3]. Thus if E is not a *KB*-space, there exists a b-bounded disjoint sequence (e_n) in E_+ with $\|e_n\| = 1$ for all n. Hence by the Lemma, there exists a positive disjoint sequence (g_n) in E' with $\|g_n\| \leqslant 1$ such that $g_n(e_n) = 1$, $g_n(e_m) = 0$ for all $n \neq m$.

We define a positive operator $T_1 : E \to l_\infty$ as follows:

$$x \to T_1(x) = (g_1(x), g_2(x), \ldots)$$

for each x in E. Let us note that $T_1(B_E) \subseteq B_{l_\infty}$.

On the other hand, since F is not a *KB*-space, we can find a b-bounded disjoint sequence in F_+ such that $0 \leqslant f_n \leqslant f$ for some f in F'' and satisfying $\|f_n\| = 1$ for all n. Let (α_n) be a positive sequence in l_∞. Then,

$$0 \leqslant \sum_{i=1}^{n} \alpha_i f_i \leqslant \sum_{i=1}^{n+1} \alpha_i f_i \leqslant \sup(\alpha_i) f$$

shows that the sequence $(\sum_{i=1}^{n} \alpha_i f_i)_n$ is an increasing norm bounded sequence in F. As F is assumed to have the Levi Property, supremum of $(\sum_{i=1}^{n} \alpha_i f_i)_n$ exists in F. We denote this supremum by $\sum_{i=1}^{\infty} \alpha_i f_i$. This enables us to define an operator $T_2 : l_\infty^+ \to F$ by $T_2(\alpha_i) = \sum_{i=1}^{\infty} \alpha_i f_i$.

T_2 has an extension to l_∞ which we will also denote by T_2.

Since (f_i) is a disjoint sequence, it follows from

$$0 \leqslant \sum_{i=1}^{n} f_i = \bigvee f_i \leqslant f$$

that $0 \leqslant (\sum_{i=1}^{n} f_i)_n$ is also an increasing norm bounded sequence in F_+. Therefore the supremum of this sequence exists in F and will be denoted by f_0. Then $T_2(B_{l_\infty}) \subseteq [-f_0, f_0]$. Now we consider the operator $T = T_2 T_1$ defined as

$$x \to \sum_{i=1}^{\infty} g_i(x) f_i$$

T is well-defined and is positive. It follows from

$$T(B_E) = T_2T_1(B_E) \subseteq T_2(B_{l_\infty}) \subseteq [-f_0, f_0]$$

that T is semicompact. However, the operator T is not b-weakly compact as

$$T(e_n) = \sum_{i=1}^{\infty} g_i(e_n) f_i = f_n$$

for all n and $\|T(e_n)\| = \|f_n\| = 1$ for all n. Recall that if T were b-weakly compact then we would have $T(e_n) \to 0$ in norm. $\triangleright$

The assumption that F has Levi Property is essential. In fact, if we take $E = l_\infty$, $F = c_0$, then each operator from E into F is weakly compact and therefore b-weakly compact. However neither E nor F is a KB-space.

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