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WEAKLY COMPACT-FRIENDLY OPERATORS[1]

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To Şafak Alpay, on the occasion of his sixtieth birthday

We introduce weak compact-friendliness as an extension of compact-friendliness, and and prove that if a non-zero weakly compact-friendly operator $B : E \to E$ on a Banach lattice is quasi-nilpotent at some non-zero positive vector, then B has a non-trivial closed invariant ideal. Relevant facts related to compact-friendliness are also discussed.

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Introduction

Throughout the paper all operators will be assumed to be one-to-one and to have dense range, since otherwise the kernel in the former case and the closure of the range in the latter case of the corresponding operator is the required non-trivial closed invariant subspace for it.

The letters X and Y denote infinite-dimensional Banach spaces while the letters E and F infinite-dimensional Banach lattices. As usual, $L(X,Y)$ stands for the algebra of all bounded linear operators between X and Y, and $L(X) := L(X,X)$. An operator $P \in L(X,Y)$ is said to be a *quasi-affinity* if P is one-to-one and has dense range. The operators $S \in L(X)$ and $T \in L(Y)$ are *quasi-similar*, denoted by $S \overset{qs}{\sim} T$, if there exist quasi-affinities $P \in L(X,Y)$ and $Q \in L(Y,X)$ such that $TP = PS$ and $QT = SQ$. The notion of quasi-similarity was first introduced by Sz.-Nagy and Foiaş in connection with their work on the harmonic analysis of operators on Hilbert space [6]. Note that quasi-similarity is an equivalence relation on the class of all operators. For more about this concept, see [4, 5].

An operator T on E is said to be *dominated* by a positive operator B on E, denoted by $T \prec B$, provided $|Tx| \leqslant B(|x|)$ for each $x \in E$. An operator which is dominated by a multiple of the identity operator is called a *central operator*. A positive operator B on E is said to be *compact-friendly* [1] if there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that R and B commute, and C is dominated by both R and K. It is worth mentioning that the notion of compact-friendliness is of substance only on infinite-dimensional Banach lattices, since every positive operator on a finite-dimensional Banach lattice is compact. Also, if B is compact, letting $R = K = C = B$ in the definition, it is seen that compact operators are compact-friendly, but the converse is not true as the identity

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operator on an infinite-dimensional space shows. Lastly, it is straightforward to observe that any power of a compact-friendly operator is also compact-friendly. A fairly complete treatment of compact-friendly operators is given in [1, 2].

For a positive operator B on a Banach lattice E, the *super right-commutant* $[B\rangle$ and the *super left-commutant* $\langle B]$ are defined, respectively, by

$$[B\rangle := \{A \in L(E)^+ \mid AB - BA \geqslant 0\} \quad \text{and} \quad \langle B] := \{A \in L(E)^+ \mid AB - BA \leqslant 0\}.$$

For all unexplained notation and terminology, we refer to [1, 2].

In analogy with the notions of quasi-similarity and compact-friendliness, we define the concepts of weak positive quasi-similarity and weak compact-friendliness.

Definition 1.1. Two positive operators $B \in L(E)$ and $T \in L(F)$ are *weakly positively quasi-similar*, denoted by $B \overset{w}{\sim} T$, if there exist positive quasi-affinities $P \in L(F, E)$ and $Q \in L(E, F)$ such that $BP \leqslant PT$ and $TQ \leqslant QB$.

Definition 1.2. A positive operator $B \in L(E)$ is called *weakly compact-friendly* if there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $R \in [B\rangle$, and C is dominated by both R and K.

The following simple but important fact is the basic structural property of weak positive quasi-similarity, and for the sake of completeness, we include it in this section.

Theorem 1.3. *$\overset{w}{\sim}$ is an equivalence relation on the class of all positive operators.*

$\lhd$ Taking $P = Q = I$, it is readily seen that $\overset{w}{\sim}$ is reflexive. As for the symmetry, the relation $B \overset{w}{\sim} T$ brings the positive quasi-affinities P and Q such that $BP \leqslant PT$ and $TQ \leqslant QB$; hence, taking $P_0 := Q$ and $Q_0 := P$, one sees that $TP_0 \leqslant P_0B$ and $BQ_0 \leqslant Q_0T$, i. e., $T \overset{w}{\sim} B$. To see that $\overset{w}{\sim}$ is transitive, assume that $B \overset{w}{\sim} T$ and $T \overset{w}{\sim} S$, whence $BP_1 \leqslant P_1T$ and $TQ_1 \leqslant Q_1B$, and $TP_2 \leqslant P_2S$ and $SQ_2 \leqslant Q_2T$ for some positive quasi-affinities P_i, Q_i $(i = 1, 2)$. Then one gets $BP_1P_2 \leqslant P_1TP_2 \leqslant P_1P_2S$ and $SQ_2Q_1 \leqslant Q_2TQ_1 \leqslant Q_2Q_1B$. Thus, $P_3 := P_1P_2$ and $Q_3 := Q_2Q_1$ are the required positive quasi-affinities to conclude that $BP_3 \leqslant P_3S$ and $SQ_3 \leqslant Q_3B$, i. e., $B \overset{w}{\sim} S$. $\rhd$

Main Results

We begin with the following fact, which will be used in the sequel and which is also of interest in itself.

Theorem 2.1. *Let B and T be two positive operators on a Banach lattice E. If B is weakly compact-friendly and is weakly positively quasi-similar to T, then T is also weakly compact-friendly.*

$\lhd$ Since $T \overset{w}{\sim} B$, there exist quasi-affinities P and Q such that $BP \leqslant PT$ and $TQ \leqslant QB$. As B is compact-friendly, there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $R \in [B\rangle$, $C \prec R$, and $C \prec K$. Therefore, it follows from $BR \leqslant RB$ that $BRP \leqslant RBP \leqslant RPT$, which implies $QBRP \leqslant QRPT$, which in turn implies that $T(QRP) \leqslant QBRP \leqslant (QRP)T$. On the other hand, the dominations $C \prec R$ and $C \prec K$ yield $QCP \prec QRP$ and $QCP \prec QKP$, respectively. It is, thus, enough to take $R_1 := QRP$, $K_1 := QKP$, and $C_1 := QCP$ as the required three operators for the weak compact-friendliness of T. $\rhd$

Remark 2.2. Theorem 2.1 shows that, although quasi-similarity doesn't preserve compactness of an operator as shown by T.B. Hoover in [4, p. 683], weak positive quasi-similarity preserves weak compact-friendliness.

Theorem 2.3. *If a non-zero weakly compact-friendly operator $B : E \to E$ on a Banach lattice is quasi-nilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal. Moreover, for each sequence $(T_n)_{n\in\mathbb{N}}$ in $[B\rangle$ there exists a non-trivial closed ideal that is invariant under B and under each T_n.*

$\lhd$ The proof follows the same lines of thought with the proof of [1, Theorem 10.55], and therefore we will only outline it. Assume, without loss of generality, that $\|B\| < 1$, and take arbitrary scalars $\alpha_n > 0$ that are small enough so that the positive operator $T := \sum_{n=1}^{\infty} \alpha_n T_n$ exists and $\|B+T\| < 1$. It then follows that $T \in [B\rangle$, and by the same reasoning, the operator $(B+T)^n$ also belongs to $[B\rangle$ for each $n \in \mathbb{N}$, whence the positive operator $A := \sum_{n=0}^{\infty}(B+T)^n$ is an element of $[B\rangle$, too. For each $x > 0$, denote by J_x the principal ideal generated by Ax, and observe that J_x is a non-zero ideal which is $(B+T)$-invariant. It then follows that J_x is invariant under B and T, and under each T_n. Hence, provided that $J_x \neq E$ for some $x > 0$, the ideal $\overline{J_x}$ is the sought one which is invariant under B and under each T_n. Thus, one can assume that $\overline{J_x} = E$ for each $x > 0$, which is the same as saying that Ax is a quasi-interior point in E for each $x > 0$. Since B is weakly compact-friendly, one can fix three non-zero operators $R, K, C : E \to E$ with R, K positive and K compact such that

$$R \in [B\rangle, \quad C \prec R, \quad \text{and} \quad C \prec K.$$

The rest of the proof uses verbatim the arguments in [1, Theorem 10.55]. $\rhd$

The following corollaries are then immediate.

Corollary 2.4. *Let B and T be two positive operators on a Banach lattice E such that B is weakly compact-friendly and T is locally quasi-nilpotent at a non-zero positive element of E. If $B \overset{w}{\sim} T$, then T has a non-trivial closed invariant subspace.*

$\lhd$ By virtue of Theorem 2.1, the locally quasi-nilpotent operator T is also weakly compact-friendly, and hence it has a non-trivial closed invariant subspace by Theorem 2.3. $\rhd$

Corollary 2.5. *Let T be a locally quasi-nilpotent operator such that there exists a non-zero positive operator in $[T\rangle$ dominated by a compact operator. Then T has a non-trivial closed invariant ideal.*

$\lhd$ Being also weakly-compact friendly by Theorem 2.1, the locally quasi-nilpotent operator T has a non-trivial closed invariant ideal by Theorem 2.3. $\rhd$

Lemma 2.6. *If a positive operator B is weakly positively-quasi similar to a compact operator, then $[B\rangle$ contains a compact operator.*

$\lhd$ Let $B \overset{w}{\sim} K$ with K compact, so that there exist positive quasi-affinities P and Q such that $BP \leqslant PK$ and $KQ \leqslant QB$. Thus, $BPKQ \leqslant PK^2Q \leqslant PKQB$, i. e., the compact operator $K_0 := PKQ$ belongs to $[B\rangle$. $\rhd$

Theorem 2.7. *Let T be a locally quasi-nilpotent positive operator which is weakly positively quasi-similar to a compact operator. Then T has a non-trivial closed invariant subspace.*

$\lhd$ By Lemma 2.6, $[T\rangle$ contains a compact operator, which is dominated by itself. Hence, Corollary 2.5 yields the existence of a T-invariant subspace. $\rhd$

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