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ON SOME PROPERTIES OF EXTENSIONS OF COMMUTATIVE UNITAL RINGS

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We find necessary and sufficient conditions for the ring $R[\alpha]$ to be either a field or a domain whenever R is a commutative ring with 1 and α is an algebraic element over R. This continues the studies started by Nachev (Compt. Rend. Acad. Bulg. Sci., 2004) and (Commun. Alg., 2005) as well as their generalization due to Mihovski (Compt. Rend. Acad. Bulg. Sci., 2005).

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1. Introduction

Throughout the text, let R be a commutative ring with identity (often called a commutative unital ring) and with multiplicative group of units R^*. Likewise, let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial of the variable x over R such that $a_0 \in R^*$. Traditionally, $R[x]$ is the ring of all polynomials of x over R; thereby $f(x) \in R[x]$. For an arbitrary but fixed element α, suppose $f(x)$ is the minimal polynomial in $R[x]$ for which $f(\alpha) = 0$, i. e. α is a root of $f(x)$. Such an $f(x)$ will be hereafter denoted by $f_\alpha(x)$.

Define

$$R[\alpha] = \left\{r_0 + r_1\alpha + \cdots + r_{n-1}\alpha^{n-1} |\, r_i \in R,\ 0 \leqslant i \leqslant n-1\right\}.$$

The algebraic operations in $R[\alpha]$ are in the usual way taking into account that $a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n = 0$ and $a_0 \in R^*$. Thus $R[\alpha]$ is a free R-module with a base

$$1, \alpha, \ldots, \alpha^{n-1}$$

as well as it is a commutative unital ring which contains α. Besides, the proper inclusion $R \subset R[\alpha]$ holds fulfilled whenever $\alpha \notin R$.

It is well known that the following module, respectively ring, isomorphism

$$R[\alpha] \cong R[x]/(f_\alpha(x))$$

holds true.

The study of $R[\alpha]$ arises quite naturally in questions concerning commutative group rings (see, e. g., [4]), where α is a fixed root of an irreducible divisor of the cyclotomic polynomial over R. The properties of $R[\alpha]$ are closely related to these of the group algebra RG where G is a finite abelian group of order n since their constructions are similar. Specifically, the construction of RG where G is cyclic of exponent n arises from that of $R[\alpha]$ for α satisfying the equation $\alpha^n = 1$.

For application purposes in this way, Nachev [5, 6] found a sufficient condition when $R[\alpha]$ does not contain nilpotents or idempotents, respectively. Note also that the polynomials in [5] and [6] are monic, that is $a_0 = 1$.

Recently, Mihovski [3] generalizes the aforementioned results of Nachev in the part for the lack of nilpotent elements in $R[\alpha]$ to an arbitrary, not necessarily monic, polynomial $f(x)$.

Here we shall explore the behavior of another specific elements in commutative rings that elements are zero divisors and units. So, the aim of the present paper is to describe by finding appropriate necessary and sufficient conditions all zero divisors and units in the algebraic ring extension $R[\alpha]$ of R, respectively, only in terms associated with R and $f_\alpha(x)$.

2. Main results

The first chief criterion is the following.

Theorem 1. *Let R, $f_\alpha(x)$ and α be as above. Then $R[\alpha]$ is a field if and only if R is a field and $f_\alpha(x)$ is irreducible over R.*

$\lhd$ As already emphasized, the ring isomorphism

$$R[\alpha] \cong R[x]/(f_\alpha(x))$$

holds fulfilled.

Moreover, it is well known that the quotient-ring $R[x]/(f_\alpha(x))$ is a field precisely when the ideal $(f_\alpha(x))$ is maximal in $R[x]$.

First of all, suppose $R[\alpha]$ is a field and assume that $0 \neq f \in R$. Since $f \in R[\alpha]$, there exists an element $u_\alpha = r_0 + r_1\alpha + \cdots + r_{n-1}\alpha^{n-1} \in R[\alpha]$ such that

$$f(r_0 + r_1\alpha + \cdots + r_{n-1}\alpha^{n-1}) = 1.$$

Furthermore, $fr_0 + fr_1\alpha + \cdots + fr_{n-1}\alpha^{n-1} = 1$, i. e. $fr_0 = 1$, $fr_1 = 0, \ldots, fr_{n-1} = 0$. Thus $f \in R^*$ and $r_1 = \cdots = r_{n-1} = 0$. Hence $u_\alpha = r_0 \in R$ and this allows us to conclude that R is really a field, as asserted. Note also the interesting fact that $(R[x])^* \cong R^*$. Nevertheless, R being a field does not imply that so is $R[x]$.

On the other hand, as already observed, the maximality of $(f_\alpha(x))$ in $R[x]$ ensures that $f_\alpha(x)$ is irreducible over R. Indeed, otherwise $f_\alpha(x) = g(x)h(x)$ for some polynomials $g(x)$ and $h(x)$ implies that $(f_\alpha(x)) \subset (g(x)) \subset R[x]$ which is wrong.

Let now $f(x)$ be irreducible over the field R. Therefore, it is easily checked that $(f_\alpha(x))$ is a maximal ideal of $R[x]$ and as we have seen this is obviously equivalent to the fact that $R[\alpha]$ is a field. In fact, since each proper ideal of $R[x]$ is a major ideal (see, for instance, [2]), it is easily verified that if $(f_\alpha(x)) \subset (g(x)) \lhd R[x]$, then $f_\alpha(x)$ must be reducible over R, which is a contradiction. $\rhd$

The next central criterion, which shows the absence of zero divisors in the ring extension $R[\alpha]$ of R via α, states like this.

Theorem 2. *For R, $f_\alpha(x)$ and α as above, $R[\alpha]$ is a domain if and only if R is a domain and $f_\alpha(x)$ is irreducible over R.*

$\lhd$ As in our preceding tactic the isomorphism of rings

$$R[\alpha] \cong R[x]/(f_\alpha(x))$$

holds valid.

Besides, it is well known that the quotient-ring $R[x]/(f_\alpha(x))$ is a domain uniquely when the ideal $(f_\alpha(x))$ is a prime ideal of $R[x]$.

Foremost, suppose $R[\alpha]$ is a domain. Hence R is a domain and as just mentioned $(f_\alpha(x))$ is a prime ideal. We claim then that $f_\alpha(x)$ is irreducible over R. To this goal, assume in a way of contradiction that $f_\alpha(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are irreducible over R (by means of a simple induction, we can restrict our attention only on two factors in the representation). Because $(f_\alpha(x))$ is a prime ideal, it follows that either $g(x) \in (f_\alpha(x))$ or $h(x) \in (f_\alpha(x))$. Consequently, letting the first relation holds, we write $g(x) = f_\alpha(x)u(x)$ for some $u(x) \in R[x]$. Thus $f_\alpha(x) = f_\alpha(x)h(x)u(x)$. But R being a domain yields that so does $R[x]$ (see, for example, [1, p. 140]). That is why, $h(x)u(x) = 1$ and $g(x) = f_\alpha(x)u(x)$ which contradicts its irreducibility. So, our assumption is false which leads to the fact that $f_\alpha(x)$ is, in fact, irreducible.

Let now $f_\alpha(x)$ be irreducible over the domain R. In order to demonstrate that $(f_\alpha(x))$ is a prime ideal in $R[x]$, assume that $a(x) \in R[x]$ and $b(x) \in R[x]$ with $a(x)b(x) \in (f_\alpha(x))$, hence $a(x)b(x) = f_\alpha(x)t(x)$ for some $t(x) \in R[x]$. Therefore, $a(\alpha)b(\alpha) = 0$. We claim that either $a(\alpha) = 0$ or $b(\alpha) = 0$. If in the contrary $a(\alpha) \neq 0$, then $f_\alpha(x)$ does not divide $a(x)$. On the other hand, the irreducible property of $f_\alpha(x)$ insures that $a(x)$ does not divide $f_\alpha(x)$ as well. This forces that $(f_\alpha(x), a(x)) = 1$ and thus there exist polynomials $v(x)$ and $w(x)$ from $R[x]$ such that $f_\alpha(x)v(x) + a(x)w(x) = 1$. Hence, we observe that $a(\alpha)w(\alpha) = 1$, whence $a(\alpha)$ is a unit in $R[\alpha]$. Thus $b(\alpha) = 0$ which substantiates our claim. Furthermore, α being a root of $b(x)$ and $f_\alpha(x)$ being minimal secure that $f_\alpha(x)/b(x)$, i. e. $b(x) \in (f_\alpha(x))$ as wanted. $\triangleright$

Remark. It is worthwhile noticing that if $f_\alpha(x)$ is irreducible over R but R is not a field, then it does not follow in general that $(f_\alpha(x))$ is maximal in $R[x]$. This is so since $R[x]$ is not a ring whose ideals are all major (compare with [2]).

In closing, we give some more comments.

Firstly, it is straightforward that $R[\alpha]$ is Noetherian if and only if R is Noetherian. In fact, it is well known that a subring and a factor-ring of a Noetherian ring is also Noetherian (e. g. [1]). Moreover, the classical Hilbert basis theorem (see cf. [1]) asserts that R being Noetherian implies that so is $R[x]$. So, the isomorphism $R[\alpha] \cong R[x]/(f_\alpha(x))$ gives our argumentation. This is a surprising fact because the property of $R[\alpha]$ being Noetherian does not depend on $f_\alpha(x)$. It will be of interest and importance to obtain such an analogous criterion for Arthinian rings. To this goal, one must characterize the structure of $R[x]$ provided that R is Arthinian. Whether or not $R[x]$ has the same property to be Arthinian?

Secondly, as it is well known and as it has been illustrated in Theorems 1 and 2 not every prime ideal is maximal. Therefore, it will be interesting to consider those class of commutative rings for which every prime ideal is maximal. An example of this matter is the situation with the so-termed *regular rings* (see [1, p. 57, Proposition 3]). These rings are defined as follows: any element $r \in R$ is said to be *a regular element* if there exists an element $u \in R$ such that $r^2u = r$. Such a complementary element u may be taken to be invertible in R, that is $u \in R^*$ (see [1, p. 58, Exercise 4]). Evidently, each idempotent is a regular element by taking $u = 1$. Thus, R is called a regular ring (named also *a von Neumann ring*) if each its element is regular, i. e. $\forall\, r \in R,\ \exists\, u \in R^* :\ r^2u = r$. Clearly, every field is a regular ring. Besides, it is also clear that each regular ring without zero divisors is a field. It is noteworthy that each regular ring is semi-simple, that is it has zero radical of Jacobson. Unfortunately, the converse affirmation is false. In fact, there is a ring without nilpotents (i. e. with zero nil-radical) which is not regular (for more details see [1, p. 57, Example]), where it is shown even that if R is regular then the ring $R[[x]]$ of all formal power series of x need not be regular as well, since there is a prime ideal in $R[[x]]$ which is not a maximal ideal; by analogy the same holds perhaps for $R[x]$. That is why, if R is a field, then $R[[x]]$ is without nilpotent elements but it is not regular. Moreover, there exists a commutative unital ring which is with no nontrivial idempotents but which contains nontrivial (i. e. different from 0 and 1) regular elements; this is the case since every unit is a regular element.

Finally, we notice that criteria when $R[\alpha]$ is a regular ring and when $R[\alpha]$ does not possess nontrivial regular elements were established by us only in terms of R and $f_\alpha(x)$. However, the complete proofs of these attainments are the theme of some other research investigation.

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