

MATHEMATICS AND ECONOMICS  
IN THE LEGACY OF LEONID KANTOROVICH

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This is a short overview of the life of Leonid Kantorovich and his contribution to the formation of the modern outlook on the interaction between mathematics and economics.

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**Kantorovich's Path**

Kantorovich was born in the family of a venereologist at St. Petersburg on January 19, 1912 (January 6, according to the old Russian style). It is curious that many reference books give another date (which is three days before). Kantorovich kept explaining with a smile that he remembers himself from January 19, 1912. The boy's talent was revealed very early. In 1926, just at the age of 14, he entered St. Petersburg (then Leningrad) State University (SPSU). Soon he started participating in a circle of G. M. Fikhtengolts for students and in a seminar on descriptive function theory. It is natural that the early academic years formed his first environment: D. K. Faddeev, I. P. Natanson, S. L. Sobolev, S. G. Mikhlin, and a few others with whom Kantorovich was friendly during all his life also participated in Fikhtengolts's circle. The old cronies called him "Lënechka" ever since these days.

After graduation from SPSU in 1930, Kantorovich started teaching, combining it with intensive scientific research. Already in 1932 he became a full professor at the Leningrad Institute of Industrial Construction Engineers and an assistant professor at SPSU. From 1934 Kantorovich was a full professor at his *alma mater*.

The main achievements in mathematics belong to the "Leningrad" period of Kantorovich's life. In the 1930s he published more papers in pure mathematics whereas his 1940s are devoted to computational mathematics in which he was soon appreciated as a leader in this country.

The letter of Academician N. N. Luzin, written on April 29, 1934, was found in the personal archive of Kantorovich a few years ago during preparation of his selected works for publication (see [1]).

This letter demonstrates the attitude of Luzin, one of the most eminent and influential mathematicians of that time, to the brilliance of the young prodigy. Luzin was the founder and leader of the famous "Lusitania" school of Muscovites. He remarked in his letter:

*... you must know my attitude to you. I do not know you as a man completely but I guess a warm and admirable personality.*

*One thing I know for certain, however: the range of your mental powers which, so far as I accustomed myself to guess people, open up limitless possibilities in science. I will*

*not utter the appropriate word — what for? Talent — this would belittle you. You are entitled to get more...*

In 1935 Kantorovich made his major mathematical discovery — he defined *K-spaces*, i.e., vector lattices whose every nonempty order bounded subset had an infimum and supremum. The Kantorovich spaces have provided the natural framework for developing the theory of linear inequalities which was a practically uncharted area of research those days. The concept of inequality is obviously relevant to approximate calculations where we are always interested in various estimates of the accuracy of results. Another challenging source of interest in linear inequalities was the stock of problems of economics. The language of partial comparison is rather natural in dealing with what is reasonable and optimal in human behavior when means and opportunities are scarce. Finally, the concept of linear inequality is inseparable from the key idea of a convex set. Functional analysis implies the existence of nontrivial continuous linear functional over the space under consideration, while the presence of a functional of this type amounts to the existence of nonempty proper open convex subset of the ambient space. Moreover, each convex set is generically the solution set of an appropriate system of simultaneous linear inequalities.

At the end of the 1940s Kantorovich formulated and explicated the thesis of interdependence between functional analysis and applied mathematics:

*There is now a tradition of viewing functional analysis as a purely theoretical discipline far removed from direct applications, a discipline which cannot deal with practical questions. This article<sup>1</sup> is an attempt to break with this tradition, at least to a certain extent, and to reveal the relationship between functional analysis and the questions of applied mathematics...*

He distinguished the three techniques: the Cauchy method of majorants also called *domination*, the method of finite-dimensional approximations, and the Lagrange method for the new optimization problems motivated by economics.

Kantorovich based his study of the Banach space versions of the Newton method on domination in general ordered vector spaces.

Approximation of infinite-dimensional spaces and operators by their finite-dimensional analogs, which is *discretization*, must be considered alongside the marvelous universal understanding of computational mathematics as the science of finite approximations to general (not necessarily metrizable) compacta.<sup>2</sup>

The novelty of the extremal problems arising in social sciences is connected with the presence of multidimensional contradictory utility functions. This raises the major problem of agreeing conflicting aims. The corresponding techniques may be viewed as an instance of *scalarization* of vector-valued targets.

From the end of the 1930s the research of Kantorovich acquired new traits in his audacious breakthrough to economics. Kantorovich's booklet *Mathematical Methods in the Organization and Planning of Production* which appeared in 1939 is a material evidence of the birth of linear programming. Linear programming is a technique of maximizing a linear functional over the positive solutions of a system of linear inequalities. It is no wonder that the discovery of linear programming was immediate after the foundation of the theory of Kantorovich spaces.

The economic works of Kantorovich were hardly visible at the surface of the scientific information flow in the 1940s. But the problems of economics prevailed in his creative studies.

<sup>1</sup> Cp. [2]; the excerpt is taken from [3: Part 2, p. 171].

<sup>2</sup> This revolutionary definition was given in the joint talk by S. L. Sobolev, L. A. Lyusternik, and L. V. Kantorovich at the Third All-Union Mathematical Congress in 1956; cp. [4, pp. 443–444].

During the Second World War he completed the first version of his book *The Best Use of Economic Resources* which led to the Nobel Prize awarded to him and Tjalling C. Koopmans in 1975.

The Council of Ministers of the USSR issued a top secret Directive No. 1990–774ss/op<sup>3</sup> in 1948 which ordered “to organize in the span of two weeks a group for computations with the staff up to 15 employees in the Leningrad Division of the Mathematical Institute of the Academy of Sciences of the USSR and to appoint Professor Kantorovich the head of the group.” That was how Kantorovich was enlisted in the squad of participants of the project of producing nuclear weapons in the USSR<sup>4</sup>.

In 1957 Kantorovich accepted the invitation to join the newly founded Siberian Division of the Academy of Sciences of the USSR. He moved to Novosibirsk and soon became a corresponding member of the Department of Economics in the first elections to the Siberian Division. Since then his major publications were devoted to economics with the exception of the celebrated course of functional analysis, “Kantorovich and Akilov” in the students’ jargon.

It is impossible not to mention one brilliant twist of mind of Kantorovich and his students in suggesting a scientific approach to taxicab metered rates. The people of the elder generation in this country remember that in the 1960s the taxicab meter rates were modernized radically: there appeared a price for taking a taxicab which was combining with a less per kilometer cost. This led immediately to raising efficiency of taxi parks as well as profitability of short taxicab drives. This economic measure was a result of a mathematical modeling of taxi park efficiency which was accomplished by Kantorovich with a group of young mathematicians and published in the rather prestigious mathematical journal *Russian Mathematical Surveys*.

The 1960s became the decade of his recognition. In 1964 he was elected a full member of the Department of Mathematics of the Academy of Sciences of the USSR, and in 1965 he was awarded the Lenin Prize. In these years he vigorously propounded and maintained his views of interplay between mathematics and economics and exerted great efforts to instill the ideas and methods of modern science into the top economic management of the Soviet Union, which was almost in vain.

At the beginning of the 1970s Kantorovich left Novosibirsk for Moscow where he was deeply engaged in economic analysis, not ceasing his efforts to influence the everyday economic practice and decision making in the national economy. His activities were mainly waste of time and stamina in view of the misunderstanding and hindrance of the governing retrogradists of this country. Cancer terminated his path in science on April 7, 1986. He was buried at Novodevichy Cemetery in Moscow.

### Contribution to Science

The scientific legacy of Kantorovich is immense. His research in the areas of functional analysis, computational mathematics, optimization, and descriptive set theory has had a dramatic impact on the foundation and progress of these disciplines. Kantorovich deserved his status of one of the father founders of the modern economic-mathematical methods. Linear programming, his most popular and celebrated discovery, has changed the image of economics.

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<sup>3</sup>The letters “ss” abbreviate the Russian for “top secret,” while the letters “op” abbreviate the Russian for “special folder.”

<sup>4</sup>This was the Soviet project “Enormous,” transliterated in Russian like “Énormoz.” The code name was used in the operative correspondence of the intelligence services of the USSR.

Kantorovich wrote more than 300 articles. When we discussed with him the first edition of an annotated bibliography of his publications in the early 1980s, he suggested to combine them in the nine sections: descriptive function theory and set theory, constructive function theory, approximate methods of analysis, functional analysis, functional analysis and applied mathematics, linear programming, hardware and software, optimal planning and optimal prices, and the economic problems of a planned economy.

Discussing the mathematical papers of Kantorovich, we must especially mention the three articles [2, 5, 6] in *Russian Mathematical Surveys*. The first of them had acquired the title that is still impressive in view of its scale, all the more if compared with the age of the author. This article appeared in the formula of the Stalin Prize of 100,000 rubles which was awarded to Kantorovich in 1948. The ideas of this brilliant masterpiece laid grounds for the classical textbook by Kantorovich and Akilov which was the deskbook of many scientists of theoretical and applied inclination.

The impressive diversity of these areas of research rests upon not only the traits of Kantorovich but also his methodological views. He always emphasized the innate integrity of his scientific research as well as mutual penetration and synthesis of the methods and techniques he used in solving the most diverse theoretic and applied problems of mathematics and economics.

The characteristic feature of the contribution of Kantorovich is his orientation to the most topical and difficult problems of mathematics and economics of his epoch.

### Functional Analysis and Applied Mathematics

The creative style of Kantorovich rested on the principle of unity of theoretical and applied studies. This principle led him to the first-rate achievements at the frontiers between functional analysis and applied mathematics. Kantorovich's technique consisted in developing and applying the methods of domination, approximation, and scalarization.

Let  $X$  and  $Y$  be real vector spaces lattice-normed with Kantorovich spaces  $E$  and  $F$ . In other words, given are some lattice-norms  $|\cdot|_X$  and  $|\cdot|_Y$ . Assume further that  $T$  is a linear operator from  $X$  to  $Y$  and  $S$  is a positive operator from  $E$  to  $F$  satisfying

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ |\cdot|_X \downarrow & & \downarrow |\cdot|_Y \\ E & \xrightarrow[S]{} & F \end{array}$$

Moreover, in case

$$|Tx|_Y \leq S|x|_X \quad (x \in X),$$

we call  $S$  the *dominant* or *majorant* of  $T$ . If the set of all dominants of  $T$  has the least element, then the latter is called the *abstract norm* or *least dominant* of  $T$  and denoted by  $|T|$ . Hence, the least dominant  $|T|$  is the least positive operator from  $E$  to  $F$  such that

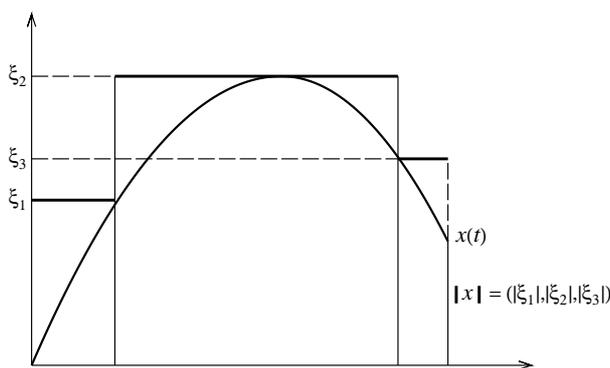
$$|Tx| \leq |T|(|x|) \quad (x \in X).$$

Kantorovich wrote about this matter in [7] as follows:

*The abstract norm enables us to estimate an element much sharper than a single number, a real norm. We can thus acquire more precise (and more broad) boundaries of the*

range of application of successive approximations. For instance, as a norm of a continuous function we can take the set of the suprema of its modulus in a few partial intervals. . . This allows us to estimate the convergence domain of successive approximations for integral equations. In the case of an infinite system of equations we know that each solution is as a sequence and we can take as the norm of a sequence not only a sole number but also finitely many numbers; for instance, the absolute values of the first entries and the estimation of the remainder:

$$\|(\xi_1, \xi_2, \dots)\| = (|\xi_1|, |\xi_2|, \dots, |\xi_{N-1}|, \sup_{k \geq N} |\xi_k|) \in \mathbb{R}^N.$$



This enables us to specify the conditions of applicability of successive approximations for infinite simultaneous equations. Also, this approach allows us to obtain approximate (surplus or deficient) solutions of the problems under consideration with simultaneous error estimation. I believe that the use of members of semiordered linear spaces instead of reals in various estimations can lead to essential improvement of the latter.

The most general domination underlaid the classical studies of Kantorovich on the Newton method which brought him international fame.

These days the development of domination proceeds within the framework of Boolean valued analysis (cp. [8]). The modern technique of mathematical modeling opened an opportunity to demonstrate that the principal properties of lattice normed spaces represent the Boolean valued interpretations of the relevant properties of classical normed spaces. The most important interrelations here are as follows: Each Banach space inside a Boolean valued model becomes a universally complete Banach–Kantorovich space in result of the external deciphering of constituents. Moreover, each lattice normed space may be realized as a dense subspace of some Banach space in an appropriate Boolean valued model. Finally, a Banach space  $X$  results from some Banach space inside a Boolean valued model by a special machinery of bounded descent if and only if  $X$  admits a complete Boolean algebra of norm-one projections which enjoys the cyclicity property. The latter amounts to the fact that  $X$  is a Banach–Kantorovich space and  $X$  is furnished with a mixed norm.<sup>5</sup>

Summarizing his research into the general theory of approximation methods, Kantorovich wrote:<sup>6</sup>

*There are many distinct methods for various classes of problems and equations, and constructing and studying them in each particular case presents considerable difficulties.*

<sup>5</sup> The modern theory of dominated operators is thoroughly set forth in the book [9] by A. G. Kusraev.

<sup>6</sup> Cp. [10].

Therefore, the idea arose of evolving a general theory that would make it possible to construct and study them with a single source. This theory was based on the idea of the connection between the given space, in which the equation to be studied is specified, and a more simple one into which the initial space is mapped. On the basis of studying the “approximate equation” in the simpler space the possibility of constructing and studying approximate methods in the initial space was discovered. . .

It seems to me that the main idea of this theory is of a general character and reflects the general gnoseological principle for studying complex systems. It was, of course, used earlier, and it is also used in systems analysis, but it does not have a rigorous mathematical apparatus. The principle consists simply in the fact that to a given large complex system in some space a simpler, smaller dimensional model in this or a simpler space is associated by means of one-to-one or one-to-many correspondence. The study of this simplified model turns out, naturally, to be simpler and more practicable. This method, of course, presents definite requirements on the quality of the approximating system.

The classical scheme of discretization as suggested by Kantorovich for the analysis of the equation  $Tx = y$ , with  $T : X \rightarrow Y$  a bounded linear operator between some Banach spaces  $X$  and  $Y$ , consists in choosing finite-dimensional approximating subspaces  $X_N$  and  $Y_N$  and the corresponding embeddings  $\iota_N$  and  $J_N$ :

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \iota_N \uparrow & & \uparrow J_N \\ X_N & \xrightarrow{T_N} & Y_N \end{array}$$

In this event, the equation

$$T_N x_N = y_N$$

is viewed as a finite-dimensional approximation to the original problem.

Boolean valued analysis enables us to expand the range of applicability of Banach–Kantorovich spaces and more general modules for studying extensional equations.

Many promising possibilities are open by the new method of *hyperapproximation* which rests on the ideas of infinitesimal analysis. The classical discretization approximates an infinite-dimensional space with the aid of finite-dimensional subspaces. Arguing within non-standard set theory we may approximate an infinite-dimensional vector space with external finite-dimensional spaces. Undoubtedly, the dimensions of these hyperapproximations are given as actually infinite numbers.

The tentative scheme of hyperapproximation is reflected by the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \varphi_E \downarrow & & \downarrow \varphi_F \\ E^\# & \xrightarrow{T^\#} & F^\# \end{array}$$

Here  $E$  and  $F$  are normed vector space over the same scalars;  $T$  is a bounded linear operator from  $E$  to  $F$ ; and  $\#$  symbolizes the taking of the relevant nonstandard hull.

Let  $E$  be an internal vector space over  ${}^*\mathbb{F}$ , where  $\mathbb{F}$  is the *basic field* of scalars; i. e., the *reals*  $\mathbb{R}$  or *complexes*  $\mathbb{C}$ , while  $*$  is the symbol of the Robinsonian standardization. Hence, we are given the two internal operations  $+$  :  $E \times E \rightarrow E$  and  $\cdot$  :  ${}^*\mathbb{F} \times E \rightarrow E$  satisfying the

usual axioms of a vector space. Since  $\mathbb{F} \subset {}^*\mathbb{F}$ , the internal vector space  $E$  is a vector space over  $\mathbb{F}$  as well. In other words,  $E$  is an external vector space that is neither a normed nor a Hilbert space externally even if  $E$  is endowed with either structure as an internal space. With each normed or pre-Hilbert space we can however associate some external Banach or Hilbert space.

Let  $(E, \|\cdot\|)$  be an internal normed space over  ${}^*\mathbb{F}$ . As usual,  $x \in E$  is a *limited* element provided that  $\|x\|$  is a limited real (whose modulus has a standard upper bound by definition). If  $\|x\|$  is an infinitesimal (= infinitely small real) then  $x$  is also referred to as an *infinitesimal*. Denote by  $\text{ltd}(E)$  and  $\mu(E)$  the external sets of limited elements and infinitesimals of  $E$ . The set  $\mu(E)$  is the *monad* of the origin in  $E$ . Clearly,  $\text{ltd}(E)$  is an external vector space over  $\mathbb{F}$ , and  $\mu(E)$  is a subspace of  $\text{ltd}(E)$ . Denote the factor-space  $\text{ltd}(E)/\mu(E)$  by  $E^\#$ . The space  $E^\#$  is endowed with the natural norm by the formula

$$\|\varphi x\| := \|x^\#\| := \text{st}(\|x\|) \in \mathbb{F} \quad (x \in \text{ltd}(E)).$$

Here  $\varphi := \varphi_E := (\cdot)^\# : \text{ltd}(E) \rightarrow E^\#$  is the canonical homomorphism, and  $\text{st}$  stands for the taking of the standard part of a limited real. In this event  $(E^\#, \|\cdot\|)$  becomes an external normed space that is called the *nonstandard hull* of  $E$ . If  $(E, \|\cdot\|)$  is a standard space then the nonstandard hull of  $E$  is by definition the space  $({}^*E)^\#$  corresponding to the Robinsonian standardization  ${}^*E$ .

If  $x \in E$  then  $\varphi({}^*x) = ({}^*x)^\#$  belongs to  $({}^*E)^\#$ . Moreover,  $\|x\| = \|({}^*x)^\#\|$ . Therefore, the mapping  $x \mapsto ({}^*x)^\#$  is an isometric embedding of  $E$  in  $({}^*E)^\#$ . It is customary to presume that  $E \subset ({}^*E)^\#$ .

Suppose now that  $E$  and  $F$  are internal normed spaces and  $T : E \rightarrow F$  is an internal bounded linear operator. The set of reals

$$c(T) := \{C \in {}^*\mathbb{R} : (\forall x \in E) \|Tx\| \leq C\|x\|\}$$

is internal and bounded. Recall that  $\|T\| := \inf c(T)$ .

If the norm  $\|T\|$  of  $T$  is limited then the classical normative inequality  $\|Tx\| \leq \|T\| \|x\|$  valid for all  $x \in E$  implies that  $T(\text{ltd}(E)) \subset \text{ltd}(F)$  and  $T(\mu(E)) \subset \mu(F)$ . Consequently, we may soundly define the descent of  $T$  to the factor space  $E^\#$  as the external operator  $T^\# : E^\# \rightarrow F^\#$ , acting by the rule

$$T^\# \varphi_E x := \varphi_F T x \quad (x \in E).$$

The operator  $T^\#$  is linear (with respect to the members of  $\mathbb{F}$ ) and bounded; moreover,  $\|T^\#\| = \text{st}(\|T\|)$ . The operator  $T^\#$  is called the *nonstandard hull* of  $T$ . It is worth noting that  $E^\#$  is automatically a Banach space for each internal (possible, incomplete) normed space  $E$ . If the internal dimension of an internal normed space  $E$  is finite then  $E$  is referred to as a *hyperfinite-dimensional* space. To each normed vector space  $E$  there is a hyperfinite-dimensional subspace  $F \subset {}^*E$  containing all standard members of the internal space  ${}^*E$ .

Infinitesimal methods also provide new schemes for hyperapproximation of general compact spaces. As an approximation to a compact space we may take an arbitrary internal subset containing all standard elements of the space under approximation. Hyperapproximation of the present day stems from Kantorovich's ideas of discretization.

It was in the 1930s that Kantorovich engrossed in the practical problems of decision making. Inspired by the ideas of functional analysis and order, Kantorovich attacked these problems in the spirit of searching for an optimal solution.

Kantorovich observed as far back as in 1948 as follows:<sup>7</sup>

*Many mathematical and practical problems lead to the necessity of finding “special” extrema. On the one hand, those are boundary extrema when some extremal value is attained at the boundary of the domain of definition of an argument. On the other hand, this is the case when the functional to be optimized is not differentiable. Many problems of these sorts are encountered in mathematics and its applications, whereas the general methods turn out ineffective in regard to the problems.*

Kantorovich was among the first scientists that formulated optimality conditions in rather general extremal problems. We view as classical his approach to the theory of optimal transport whose center is occupied by the Monge–Kantorovich problem; cp. [11].

Another particularity of the extremal problems stemming from praxis consists in the presence of numerous conflicting ends and interests which are to be harmonized. In fact, we encounter the instances of multicriteria optimization whose characteristic feature is a vector-valued target. Seeking for an optimal solution in these circumstances, we must take into account various contradictory preferences which combine into a sole compound aim. Furthermore, it is impossible as a rule to distinguish some particular scalar target and ignore the rest of the targets without distorting the original statement of the problem under study.

The specific difficulties of practical problems and the necessity of reducing them to numerical calculations let Kantorovich to pondering over the nature of the reals. He viewed the members of his  $K$ -spaces as generalized numbers, developing the ideas that are now collected around the concept of scalarization.

In the most general sense, scalarization is reduction to numbers. Since number is a measure of quantity; therefore, the idea of scalarization is of importance to mathematics in general. Kantorovich’s studies on scalarization were primarily connected with the problems of economics he was interested in from the very first days of his creative path in science.

### Mathematics and Economics

Mathematics studies the forms of reasoning. The subject of economics is the circumstances of human behavior. Mathematics is abstract and substantive, and the professional decisions of mathematicians do not interfere with the life routine of individuals. Economics is concrete and declarative, and the practical exercises of economists change the life of individuals substantially. The aim of mathematics consists in impeccable truths and methods for acquiring them. The aim of economics is the well-being of an individual and the way of achieving it. Mathematics never intervenes into the private life of an individual. Economics touches his purse and bag. Immense is the list of striking differences between mathematics and economics.

Mathematical economics is an innovation of the twentieth century. It is then when the understanding appeared that the problems of economics need a completely new mathematical technique.

*Homo sapiens* has always been and will stay forever *homo economicus*. Practical economics for everyone as well as their ancestors is the arena of common sense. Common sense is a specific ability of a human to instantaneous moral judgement. Understanding is higher than common sense and reveals itself as the adaptability of behavior. Understanding is not

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<sup>7</sup> Cp. [7].

inherited and so it does not belong to the inborn traits of a person. The unique particularity of humans is the ability of sharing their understanding, transforming evaluations into material and ideal artefacts.

Culture is the treasure-trove of understanding. The inventory of culture is the essence of outlook. Common sense is subjective and affine to the divine revelation of faith that is the force surpassing the power of external proofs by fact and formal logic. The verification of statements with facts and by logic is a critical process liberating a human from the errors of subjectivity. Science is an unpaved road to objective understanding. The religious and scientific versions of outlook differ actually in the methods of codifying the artefacts of understanding.

The rise of science as an instrument of understanding is a long and complicated process. The birth of ordinal counting is fixed with the palaeolithic findings that are separated by hundreds of centuries from the appearance of a knowing and economic human. Economic practice precedes the prehistory of mathematics which became the science of provable calculations in Ancient Greece about 2500 years ago.

It was rather recently that the purposeful behavior of humans under the conditions of limited resources became the object of science. The generally accepted date of the birth of economics as a science is March 9, 1776 — the day when there was published the famous book by Adam Smith *An Inquiry into the Nature and Causes of the Wealth of Nations*.

### Consolidation of Mind

Ideas rule the world. John Maynard Keynes completed this banal statement with a touch of bitter irony. He finished his most acclaimed treatise *The General Theory of Employment, Interest, and Money* in a rather aphoristic manner:

*Practical men, who believe themselves to be quite exempt from any intellectual influences, are usually the slaves of some defunct economist.*

Political ideas aim at power, whereas economic ideas aim at freedom from any power. Political economy is inseparable from not only the economic practice but also the practical policy. The political content of economic teachings implies their special location within the world science. Changes in epochs, including their technological achievements and political utilities, lead to the universal proliferation of spread of the emotional attitude to economic theories, which drives economics in the position unbelievable for the other sciences. Alongside noble reasons for that, there is one rather cynical: although the achievements of exact sciences drastically change the life of the mankind, they never touch the common mentality of humans as vividly and sharply as any statement about their purses and limitations of freedom.

Science is “supersensible,” implying that its content cannot be wholly revealed without humans. Located in the very center of culture, science reminds of the Tower of Babel, the naive but heroic and grandiose project of the peoples of the Earth. Drive to freedom, innate in humans, lives in the unsatisfiable striving for knowledge. “We must know, we will know” — this centenarian motto of David Hilbert resides comfortably in the treasure-trove of common sense.

Georg Cantor, the creator of set theory, remarked as far back as in 1883 that “the essence of mathematics lies entirely in its freedom.” The freedom of mathematics does not reduce to the absence of exogenic restriction on the objects and methods of research. The freedom of mathematics reveals itself mostly in the new intellectual tools for conquering the ambient universe which are provided by mathematics for liberation of humans by widening the frontiers

of their independence. Mathematization of economics is the unavoidable stage of the journey of the mankind into the realm of freedom.

The nineteenth century is marked with the first attempts at applying mathematical methods to economics in the research by Antoine Augustin Cournot, Karl Marx, William Stanley Jevons, Léon Walras, and his successor in Lausanne University Vilfredo Pareto.

John von Neumann and Leonid Kantorovich, mathematicians of the first calibre, addressed the economic problems in the twentieth century. The former developed game theory, making it an apparatus for the study of economic behavior. The latter invented linear programming for decision making in the problems of best use of limited resources. These contributions of von Neumann and Kantorovich occupy an exceptional place in science. They demonstrated that the modern mathematics opens up broad opportunities for economic analysis of practical problems. Economics has been drifted closer to mathematics. Still remaining a humanitarian science, it mathematizes rapidly, demonstrating high self-criticism and an extraordinary ability of objective thinking.

The turn in the mentality of the mankind that was effected by von Neumann and Kantorovich is not always comprehended to full extent. There are principal distinctions between the exact and humanitarian styles of thinking. Humans are prone to reasoning by analogy and using incomplete induction, which invokes the illusion of the universal value of the tricks we are accustomed to. The differences in scientific technologies are not distinguished overtly, which in turn contributes to self-isolation and deterioration of the vast sections of science.

The methodological precipice between economists and mathematics was well described by Alfred Marshall, the founder of the Cambridge school of neoclassicals, “Marshallians.” He wrote in his *magnum opus* [12]:

*The function then of analysis and deduction in economics is not to forge a few long chains of reasoning, but to forge rightly many short chains and single connecting links. . .*<sup>8</sup>

*It is obvious that there is no room in economics for long trains of deductive reasoning.*<sup>9</sup>

In 1906 Marshall formulated his scepticism in regard to mathematics as follows:

*[I had] a growing feeling in the later years of my work at the subject that a good mathematical theorem dealing with economic hypotheses was very unlikely to be good economics: and I went more and more on the rules —*

- (1) *Use mathematics as a shorthand language, rather than an engine of inquiry.*
- (2) *Keep to them till you have done.*
- (3) *Translate into English.*
- (4) *Then illustrate by examples that are important in real life.*
- (5) *Burn the mathematics.*
- (6) *If you can't succeed in (4), burn (3). This last I did often.*

*I don't mind the mathematics, it's useful and necessary, but it's too bad the history of economic thought is no longer required or even offered in many graduate and undergraduate programs. That's a loss.*<sup>10</sup>

Marshall intentionally counterposed the economic and mathematical ways of thinking, noting that the numerous short “combs” are appropriate in a concrete economic analysis. Clearly, the image of a “comb” has nothing in common with the upside-down pyramid, the cumulative hierarchy of the von Neumann universe, the residence of the modern Zermelo–Fraenkel set

<sup>8</sup> Cp. [12, Appendix C: The Scope and Method of Economics. § 3].

<sup>9</sup> Cp. [12, Appendix D: Use of Abstract Reasoning in Economics].

<sup>10</sup> Cp. [13, p. 294].

theory. It is from the times of Hellas that the beauty and power of mathematics rest on the axiomatic method which presumes the derivation of new facts by however lengthy chains of formal implications.

The conspicuous discrepancy between economists and mathematicians in mentality has hindered their mutual understanding and cooperation. Many partitions, invisible but ubiquitous, were erected in ratiocination, isolating the economic community from its mathematical counterpart and vice versa.

This status quo with deep roots in history was always a challenge to Kantorovich, contradicting his views of interaction between mathematics and economics.

### Linear Programming

The principal discovery of Kantorovich at the junction of mathematics and economics is linear programming which is now studied by hundreds of thousands of people throughout the world. The term signifies the colossal area of science which is allotted to linear optimization models. In other words, linear programming is the science of the theoretical and numerical analysis of the problems in which we seek for an optimal (i. e., maximum or minimum) value of some system of indices of a process whose behavior is described by simultaneous linear inequalities.

The term “linear programming” was minted in 1951 by Koopmans. The most commendable contribution of Koopmans was the ardent promotion of the methods of linear programming and the strong defence of Kantorovich’s priority in the invention of these methods.

In the USA the independent research into linear optimization models was started only in 1947 by George B. Dantzig who convincingly described the history of the area in his classical book [14, p. 22–23] as follows:

*The Russian mathematician L. V. Kantorovich has for a number of years been interested in the application of mathematics to programming problems. He published an extensive monograph in 1939 entitled *Mathematical Methods in the Organization and Planning of Production*. . .*<sup>11</sup>

*Kantorovich should be credited with being the first to recognize that certain important broad classes of production problems had well-defined mathematical structures which, he believed, were amenable to practical numerical evaluation and could be numerically solved.*

*In the first part of his work Kantorovich is concerned with what we now call the weighted two-index distribution problems. These were generalized first to include a single linear side condition, then a class of problems with processes having several simultaneous outputs (mathematically the latter is equivalent to a general linear program). He outlined a solution approach based on having on hand an initial feasible solution to the dual. (For the particular problems studied, the latter did not present any difficulty.) Although the dual variables were not called “prices,” the general idea is that the assigned values of these “resolving multipliers” for resources in short supply can be increased to a point where it pays to shift to resources that are in surplus. Kantorovich showed on simple examples how to make the shifts to surplus resources. In general, however, how to shift turns out to be a linear program in itself for which no computational method was given. The report contains an outstanding collection of potential applications. . .*

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<sup>11</sup> The English translation of the book appeared as “Mathematical methods of production planning and organization” in: *The Use of Mathematics in Economics*. Edinburgh: Oliver and Boyd Limited, 1964.— P. 225–279.

*If Kantorovich's earlier efforts had been appreciated at the time they were first presented, it is possible that linear programming would be more advanced today. However, his early work in this field remained unknown both in the Soviet Union and elsewhere for nearly two decades while linear programming became a highly developed art.*

It is worth observing that to an optimal plan of every linear program there corresponds some optimal prices or “objectively determined estimators.” Kantorovich invented this bulky term by tactical reasons in order to enhance the “criticism endurance” of the concept.

The interdependence of optimal solutions and optimal prices is the crux of the economic discovery of Kantorovich.

### Universal Heuristics

The integrity of the outlook of Kantorovich was revealed in all instances of his versatile research. The ideas of linear programming were tightly interwoven with his methodological standpoints in the realm of mathematics. Kantorovich viewed as his main achievement in this area the distinguishing of  $K$ -spaces.<sup>11</sup>

Kantorovich observed in his first short paper of 1935 in *Doklady* on the newly-born area of ordered vector spaces:<sup>12</sup>

*In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in these spaces) in the same way as linear functionals.*

This was the first formulation of the most important methodological position that is now referred to *Kantorovich's heuristic principle*. It is worth noting that his definition of a semiordered linear space contains the axiom of Dedekind completeness which was denoted by  $I_6$ . Kantorovich demonstrated the role of  $K$ -spaces by widening the scope of the Hahn–Banach Theorem. The heuristic principle turned out applicable to this fundamental Dominated Extension Theorem; i. e., we may abstract the Hahn–Banach Theorem on substituting the elements of an arbitrary  $K$ -space for reals and replacing linear functionals with operators acting into the space.

Attempts at formalizing Kantorovich's heuristic principle started in the middle of the twentieth century at the initial stages of  $K$ -space theory and yielded the so-called identity preservation theorems. They assert that if some algebraic proposition with finitely many function variables is satisfied by the assignment of all real values then it remains valid after replacement of reals with members of an arbitrary  $K$ -space.

Unfortunately, no satisfactory explanation was suggested for the internal mechanism behind the phenomenon of identity preservation. Rather obscure remained the limits on the heuristic transfer principle. The same applies to the general reasons for similarity and parallelism between the reals and their analogs in  $K$ -space which reveal themselves every now and then.

The abstract theory of  $K$ -spaces, linear programming, and approximate methods of analysis were particular outputs of Kantorovich's universal heuristics. In his last mathematical paper [15] which was written when he had been mortally ill, Kantorovich remarked:

*One aspect of reality was temporarily omitted in the development of the theory of function spaces. Of great importance is the relation of comparison between practical objects,*

<sup>11</sup> Kantorovich wrote about “my spaces” in his personal memos.

<sup>12</sup> Cp. [3, Part 2, pp. 213–216].

alongside algebraic and other relations between them. Simple comparison applicable to every pair of objects is of a depleted character; for instance, we may order all items by weight which is of little avail. That type of ordering is more natural which is defined or distinguished when this is reasonable and which is left indefinite otherwise (*partial ordering or semiorde*). For instance, two sets of goods must undoubtedly be considered as comparable and one greater than the other if each item of the former set is quantitatively greater than its counterpart in the latter. If some part of the goods of one set is greater and another part is less than the corresponding part of the other then we can avoid prescribing any order between these sets. It is with this in mind that the theory of ordered vector spaces was propounded and, in particular, the theory of the above-defined  $K$ -spaces. It found various applications not only in the theoretic problems of analysis but also in construction of some applied methods, for instance the theory of majorants in connection with the study of successive approximations. At the same time the opportunities it offers are not revealed fully yet. The importance for economics is underestimated of this branch of functional analysis. But the comparison and correspondence relations play an extraordinary role in economics and it was definitely clear even at the cradle of  $K$ -spaces that they will find their place in economic analysis and yield luscious fruits.

The theory of  $K$ -spaces has another important feature: their elements can be treated as numbers. In particular, we may use elements of such a space, finite- or infinite-dimensional, as a norm in construction of analogs of Banach spaces. This choice of norms for objects is much more accurate. Say, a function is normed not by its maximum on the whole interval but a dozen of numbers, its maxima on parts of this interval.

More recent research has corroborated that the ideas of linear programming are immanent in the theory of  $K$ -spaces. It was demonstrated that the validity of one of the various statements of the duality principle of linear programming in an abstract mathematical structure implies with necessity that the structure under consideration is in fact a  $K$ -space.

The Kantorovich heuristic principle is connected with one of the most brilliant pages of the mathematics of the twentieth century — the famous problem of the continuum. Recall that some set  $A$  has the cardinality of the continuum whenever  $A$  is equipollent with a segment of the real axis. The continuum hypothesis is that each subset of the segment is either countable or has the cardinality of the continuum. The continuum problem asks whether the continuum hypothesis is true or false.

The continuum hypothesis was first conjectured by Cantor in 1878. He was convinced that the hypothesis was a theorem and vainly attempted at proving it during his whole life. In 1900 the Second Congress of Mathematicians took place in Paris. At the opening session Hilbert delivered his epoch-making talk “Mathematical Problems.” He raised 23 problems whose solution was the task of the nineteenth century bequeathed to the twentieth century. The first on the Hilbert list was open the continuum problem. Remaining unsolved for decades, it gave rise to deep foundational studies. The efforts of more than a half-century yielded the solution: we know now that the continuum hypothesis can neither be proved nor refuted.

The two stages led to the understanding that the continuum hypothesis is an independent axiom. Gödel showed in 1939 that the continuum hypothesis is consistent with the axioms of set theory<sup>13</sup>, and Cohen demonstrated in 1963 that the negation of the continuum hypothesis does not contradict the axioms of set theory either. Both results were established by exhibiting appropriate models; i. e., constructing a universe and interpreting set theory in

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<sup>13</sup> This was done for ZFC.

the universe. The Gödel approach based on “truncating” the von Neumann universe. Gödel proved that the constructible sets he distinguished yield the model that satisfies the continuum hypothesis. Therefore, the negation of the continuum hypothesis is not provable. The approach by Cohen was in a sense opposite to that of Gödel: it used a controlled enrichment of the von Neumann universe.

Cohen’s method of forcing was simplified in 1965 on using the tools of Boolean algebra and the new technique of mathematical modeling which is based on the nonstandard models of set theory. The progress of the evoked Boolean valued analysis has demonstrated the fundamental importance of the so-called universally complete  $K$ -spaces. Each of these spaces turns out to present one of the possible noble models of the real axis and so such a space plays a similar key role in mathematics. The spaces of Kantorovich implement new models of the reals, this earning their eternal immortality.

Kantorovich heuristics has received brilliant corroboration, this proving the integrity of science and inevitability of interpenetration of mathematics and economics.

### Mememes for the Future

The contradistinction between the brilliant achievements and the childish unfitnes for the practical seamy side of life is listed among the dramatic enigmas by Kantorovich. His life became a fabulous and puzzling humanitarian phenomenon. Kantorovich’s introvertnes, obvious in personal communications, was inexplicably accompanied by outright public extravertnes. The absence of any orator’s abilities neighbored his deep logic and special mastery in polemics. His innate freedom and self-sufficiency coexisted with the purposeful and indefatigable endurance in the case of necessity. He bequeathed us a magnificent example of the best use of personal resources in the presence of restrictive internal and external constraints.

The mememes of Kantorovich have been received as witnessed by the curricula and syllabi of every economics or mathematics department in any major university throughout the world. The gadgets of mathematics and the idea of optimality belong to the tool-kit of any practicing economist. The new methods erected an unsurmountable firewall against the traditionalists that view economics as a testing polygon for the technologies like Machiavellianism, flattery, common sense, or foresight.

Economics as an eternal boon companion of mathematics will avoid merging into any esoteric part of the humanities, or politics, or belles-lettres. The new generations of mathematicians will treat the puzzling problems of economics as an inexhaustible source of inspiration and an attractive arena for applying and refining their impeccably rigorous methods.

Calculation will supersede prophesy.

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МАТЕМАТИКА И ЭКОНОМИКА  
В НАСЛЕДИИ ЛЕОНИДА КАНТОРОВИЧА

Кутателадзе С. С.

Краткий обзор биографии и научного вклада Л. В. Канторовича (1912–1986), выдающегося математика и экономиста.

**Ключевые слова:** функциональный анализ, линейное программирование, математическая экономика.