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KANTOROVICH'S PRINCIPLE IN ACTION: AW^*-MODULES AND INJECTIVE BANACH LATTICES

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Making use of Boolean valued representation it is proved that Kaplansky–Hilbert lattices and injective Banach lattices may be produced from each other by means of the convexification procedure. The relationship between the Kantorovich's heuristic principle and the Boolean value transfer principle is also discussed.

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1. Introduction

The aim of this note is to demonstrate that Kaplansky–Hilbert lattices and injective Banach lattices may be produced from each other by means of the well known convexification procedure. This is done via the Boolean valued analysis approach. The subject gives a good opportunity to discuss also the relationship between the Kantorovich's heuristic principle and the Boolean value transfer principle.

Everywhere below $\mathbb{B}$ is a complete Boolean algebra and $\mathrm{V}^{(\mathbb{B})}$ the corresponding Boolean valued model of set theory, see [3, 22]. Let Λ be a real Dedekind complete AM-space with unit $\mathbb{1}$ endowed with a unique f-algebra multiplication. Then $\bar{\Lambda} := \Lambda \oplus i\Lambda$ is a commutative C^*-algebra often called a *Stone algebra*. We write $\Lambda = \Lambda(\mathbb{B})$ whenever $\mathbb{B}$ is a Boolean algebra of band projections in Λ. The unexplained terms of use below can be found in [19] and [27].

2. Kantorovich's Principle

L. V. Kantorovich was among the first who studied operators in ordered vector spaces. He indicated an important instance of ordered vector spaces, a Dedekind complete vector lattice, often called a Kantorovich space or a K-space. This notion appeared in Kantorovich's first fundamental article [16] on this topic where he wrote:

"*В этой заметке я определяю новый тип пространств, которые я называю линейными полуупорядоченными пространствами. Введение этих пространств позволяет изучать линейные операции одного общего класса (операции, значения которых принадлежат такому пространству) как линейные функционалы.*"[1]

[1] In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals.

Here Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces, claiming that the elements of a K-space can be considered as generalized reals. Essentially, this principle turned out to be one of those profound ideas that, playing an active and leading role in the formation of a new branch of analysis, led eventually to a deep and elegant theory of K-space rich in various applications. At the very beginning of the development of the theory, attempts were made at formalizing the above heuristic argument. In this direction, there appeared the so-called identity preservation theorems which claimed that if some proposition involving finitely many functional relations is proven for the reals then an analogous fact remains valid automatically for the elements of every K-space (see [17, 27, 35]). The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis. More about the Kantorovich's universal heuristics and innate integrity of his methodology see in [24].

3. Boolean Valued Analysis

Boolean valued analysis signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different Boolean valued models of set theory. As the models, we usually take the *von Neumann universe* V (the mundane embodiment of the classical Cantorian paradise) and the *Boolean valued universe* $\mathrm{V}^{(\mathbb{B})}$ (a specially-trimmed universe whose construction utilizes a complete Boolean algebra $\mathbb{B}$). The principal difference between V and $\mathrm{V}^{(\mathbb{B})}$ is the way of verification of statements: there is a natural way of assigning to each statement ϕ about $x_1, \dots, x_n \in \mathrm{V}^{(\mathbb{B})}$ the *'Boolean truth-value'* $[\![\phi(x_1, \dots, x_n)]\!] \in \mathbb{B}$. The sentence $\phi(x_1, \dots, x_n)$ is called true in $\mathrm{V}^{(\mathbb{B})}$ if $[\![\phi(x_1, \dots, x_n)]\!] = \mathbb{1}$. For any complete Boolean algebra $\mathbb{B}$, all the theorems of Zermelo–Fraenkel set theory are true in $\mathrm{V}^{(\mathbb{B})}$. There is a smooth mathematical technique for revealing interplay between the interpretations of one and the same fact in the two models V and $\mathrm{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending machinery* rests on the functors of *canonical embedding* $X \mapsto X^{\wedge}$, *descent* $X \mapsto X{\downarrow}$, and *ascent* $X \mapsto X{\uparrow}$, see [22, 23].

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis.

According to the principles of Boolean valued analysis there exists an internal field of reals $\mathscr{R}$ in the model $\mathrm{V}^{(\mathbb{B})}$ which is unique up to isomorphism. In other words, there exists $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals $]\!] = \mathbb{1}$. Moreover, if $[\![\, \mathscr{R}'$ is a field of reals $]\!] = \mathbb{1}$ for some $\mathscr{R}' \in \mathrm{V}^{(\mathbb{B})}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$.

By the same reasons there exists an internal field of complex numbers $\mathscr{C} \in \mathrm{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathrm{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathrm{V}^{(\mathbb{B})}$.

The fundamental result of Boolean valued analysis is Gordon's Theorem [10] which reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* Formally:

Theorem 1. *Let $\mathscr{R}$ be the reals inside $\mathrm{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$, with the descended operations and order, is a universally complete vector lattice. Moreover, there exists an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra of band projections in $\mathscr{R}{\downarrow}$ such that*

$$\chi(b)x = \chi(b)y \iff b \leqslant [\![\, x = y \,]\!], \quad \chi(b)x \leqslant \chi(b)y \iff b \leqslant [\![\, x \leqslant y \,]\!]$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$. Moreover, $\mathrm{V}^{(\mathbb{B})} \models$ "$\mathbb{R}^{\wedge}$ is a dense ordered subfield of $\mathscr{R}$".

The converse is also true: *Each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals* (*viewed as such over some dense subfield of the reals*). More details can be found in [19, 22, 23].

Remark 1. Applications of Boolean-valued models to functional analysis stem from the works by E. I. Gordon [10, 11] and G. Takeuti [32]. The term Boolean valued analysis is due to G. Takeuti [33, 34].

4. Kapansky–Hilbert Modules

Let X be a unitary $\bar{\Lambda}$-module. The mapping $\langle\cdot\,|\,\cdot\rangle : X \times X \to \bar{\Lambda}$ is called a $\bar{\Lambda}$-*valued inner product*, whenever for all $x, y, z \in X$ and $a \in \bar{\Lambda}$ the following are satisfied:

(1) $\langle x\,|\,x\rangle \geqslant 0$; $\langle x\,|\,x\rangle = 0 \ \Leftrightarrow\ x = 0$;

(2) $\langle x\,|\,y\rangle = \langle y\,|\,x\rangle^*$;

(3) $\langle ax\,|\,y\rangle = a\langle x\,|\,y\rangle$;

(4) $\langle x + y\,|\,z\rangle = \langle x\,|\,z\rangle + \langle y\,|\,z\rangle$.

If X is complete with respect to the norm $\|x\| := \sqrt{\|\langle x, x\rangle\|_\infty}$ $(x \in X)$, it is called a C^*-*module* over $\bar{\Lambda}$. A C^*-module X over $\bar{\Lambda}$ is a *Kaplansky–Hilbert module over* $\bar{\Lambda} = \bar{\Lambda}(\mathbb{B})$ if it enjoys the property: Given a norm-bounded family $(x_\xi)_{\xi\in\Xi}$ in X and a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$, there exists an element $x \in X$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$, see [19, Definition 7.4.5.].

Consider a Kaplansky–Hilbert module X with a $\bar{\Lambda}$-valued inner product $\langle\cdot,\cdot\rangle$. The norm $\|x\| := \sqrt{\|\langle x|x\rangle\|_\infty}$ $(x \in X)$ and the Λ-valued norm $|x| := \sqrt{\langle x|x\rangle}$ $(x \in X)$ in X are related as $\|x\| = \big\||x|\big\|_\infty$ $(x \in X)$. Moreover, two forms of the Cauchy–Bunyakovskiĭ inequality are fulfilled:

$$\langle x\,|\,y\rangle \leqslant |x| \cdot |y|, \quad \|\langle x\,|\,y\rangle\|_\infty \leqslant \|x\|\,\|y\| \quad (x, y \in X).$$

The following result due to M. Ozawa [29] (together with the other results from [28, 30]) tells us that the category of Kaplansky–Hilbert modules over $\bar{\Lambda} = \bar{\Lambda}(\mathbb{B})$ and bounded $\bar{\Lambda}$-linear operators is equivalent to the category of Hilbert spaces and bounded linear operators in $\mathrm{V}^{(\mathbb{B})}$. For a Banach space $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ the *descent* $\mathscr{X}{\downarrow}$ and the *bounded descent* $\mathscr{X}{\Downarrow}$ are defined as $\mathscr{X}{\downarrow} := \{x \in \mathrm{V}^{(\mathbb{B})} : [\![x \in \mathscr{X}]\!] = \mathbb{1}\}$ and $\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : [\![\|x\| \leqslant C^{\wedge}]\!] = \mathbb{1}$ for some $C \in \mathbb{R}_+\}$. More details see in [19, Chapter 8].

Theorem 2. *The bounded descent of an arbitrary Hilbert space in* $\mathrm{V}^{(\mathbb{B})}$ *is a Kaplansky–Hilbert module over the Stone algebra* $\bar{\Lambda}(\mathbb{B})$. *Conversely, if X is a Kaplansky–Hilbert module over* $\bar{\Lambda}(\mathbb{B})$, *then there is a Hilbert space* $\mathscr{X}$ *in* $\mathrm{V}^{(\mathbb{B})}$ *whose bounded descent* $\mathscr{X}{\Downarrow}$ *is unitarily equivalent with X. The space* $\mathscr{X}$ *is unique to within unitary equivalence inside* $\mathrm{V}^{(\mathbb{B})}$.

Remark 2. The concept of Kaplansky–Hilbert module was introduced by I. Kaplansky in [18] under the name AW^*-*module*. In the introduction he wrote:

"*. . . the new idea is to generalize Hilbert space by allowing the inner product to take values in a more general ring then the complex numbers. After the appropriate preliminary theory of these AW^*-modules has been developed, one can operate with a general AW^*-algebra of type I in almost the same manner as with the factor.*"

In other words, the central elements of an AW^*-algebra can be taken as complex numbers and one can work with factors rather then with general AW^*-algebras. Needles to say, this is a version of Kantorovich's heuristic principle.

5. Injective Banach Lattices

A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ with $\|T_0\| = \|T\|$. Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a positive contractive projection from Y onto X. Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [2, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms. More details see in Lotz [26], Cartwright [9], Haydon [14], and Buskes [5].

A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^{\perp} x\|\}$ for all $x \in X$, where $\pi^{\perp} := I_X - \pi$. The collection of all M-projections forms a subalgebra $\mathbb{M}(X)$ of the Boolean algebra of all band projections $\mathbb{P}(X)$ in X. The f-subalgebrs of the center $\mathscr{Z}(X)$ generated by $\mathbb{M}(X)$ is called the *M-center* and denoted by $\mathscr{Z}_m(X)$. Observe that the relations $\mathbb{B} \simeq \mathbb{M}(X)$ and $\Lambda(\mathbb{B}) \simeq \mathscr{Z}_m(X)$ are equivalent. The notion of an M-projection plays a crucial role in the theory of injective Banach lattices. In a wider context of a general Banach space theory the concept see in [4] and [13].

Let X and Y be Banach lattices and $\mathbb{B}$ a Boolean algebra which is identified with a subalgebra of $\mathbb{P}(X)$ and a subalgebra of $\mathbb{P}(Y)$. An operator $T : X \to Y$ is called $\mathbb{B}$-*linear*, if it is linear and $b \circ T = T \circ b$ for all $b \in \mathbb{B}$. Say that X is *lattice* $\mathbb{B}$-*isometric* to Y and write $X \simeq_{\mathbb{B}} Y$ if there is a $\mathbb{B}$-linear lattice isometry from X onto Y.

Now we are able to state a Boolean valued transfer principle from AL-spaces to injective Banach lattices. See [21] for details.

Theorem 3. *The bounded descent $X := \mathscr{X}\Downarrow$ of an AL-space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ is an injective Banach lattice with $\mathbb{B} \simeq \mathbb{M}(X)$ and $\Lambda(\mathbb{B}) \simeq \mathscr{Z}_m(X)$. Conversely, if X is an injective Banach lattice and $\mathbb{B} = \mathbb{M}(X)$, then there exists a unique up to lattice isometry AL-space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ whose bounded descent is lattice $\mathbb{B}$-isometric to X.*

Remark 3. Again Kantorovich's principle works: The M-center of an injective Banach lattice can be taken as the field of reals, since the only injective Banach lattices with one-dimensional M-centers are AL-spaces [14, Theorem 3F]. More precisely, according to Theorem 3 and principles of Boolean valued analysis, each theorem about the AL-space within Zermelo–Fraenkel set theory has an analog for the original injective Banach lattice interpreted as the Boolean-valued AL-space. Translation of theorems from AL-spaces to injective Banach lattices is carried out by appropriate general operations of Boolean-valued analysis, see [21].

6. Interaction: Kaplansky–Hilbert Lattices and Injectives

A real Banach lattice X is said to be a *Kaplansky–Hilbert lattice* over Λ whenever $X \oplus iX$ is a Kaplansky–Hilbert module over $\bar{\Lambda}$ with respect to the norm $\|x + iy\| := \sqrt{\|x\|^2 + \|y\|^2}$ $(x, y \in X)$. A Kaplansky–Hilbert lattice over $\Lambda = \mathbb{R}$ is called a *Hilbert lattice,* see [27]. Kaplansky–Hilbert lattices and injective Banach lattices are closely related and one can be transformed into another by means of the well known procedure of *α-convexification.* This surprising fact is almost trivial inside an appropriate Boolean valued model.

For $\alpha, s, t \in \mathbb{R}$, $\alpha > 0$, we denote $t^{\alpha} := \mathrm{sgn}(t)|t|^{\alpha}$ and $\sigma_{\alpha}(s,t) := (s^{1/\alpha} + t^{1/\alpha})^{\alpha}$. In a vector lattice X, we introduce new vector operations $\oplus$ and $*$, while the original ordering $\leqslant$ remain unchanged: $x \oplus y := (x^{1/\alpha} + y^{1/\alpha})^{\alpha}$, $\lambda * x := \lambda^{\alpha} x$ $(x, y \in X;\ \lambda \in \mathbb{R})$. Then

$X^{(\alpha)} := (X, \oplus, *, \leqslant)$ is again a vector lattice called an *α-convexification* of X. Note, that $1/\alpha$-covexification is also called an *α-concavification*, see [25, pp. 53, 54]. Define also a homogeneous function $\|\cdot\|_\alpha : X^{(\alpha)} \to \mathbb{R}$ by $\|x\|_\alpha := \|x\|^{1/\alpha}$. In the case of a function space X we have $X^{(\alpha)} = \{f : \ f^\alpha \in X\}$.

We need one more useful concept introduced by G. Buskes and A. van Rooij [8]. Let X be a vector lattice. The pair $(X^{\odot}, \odot)$ is called a *square* of E if the following conditions are fulfilled: (1) $X^{\odot}$ is a vector lattice; (2) $\odot : X \times X \to X^{\odot}$ is a symmetric lattice bimorphism; (3) for any vector lattice Y and every symmetric lattice bimorphism $\varphi : X \times X \to Y$ there exists a unique lattice homomorphism $S : X^{\odot} \to Y$ such that $S \circ \odot = \varphi$.

Theorem 4. *An arbitrary Archimedean vector lattice X has a unique (up to a lattice isomorphism) square $(X^{\odot}, \odot)$. If X is uniformly complete then $X^{\odot} = X^{(1/2)}$ and $x \odot y := (xy)^{1/2}$ for all $x, y \in X$. If X is a q-convex Banach lattice for some $q \geqslant 2$, then $X^{\odot}$ equipped with the norm $\|x \odot x\|^{\odot} := \|x \odot x\|_{1/2} := \|x\|^2$ is also a Banach lattice.*

$\lhd$ The existence of $X^{\odot}$ was established in [8]. For the identity $X^{\odot} = X^{(1/2)}$ see [31, Proposition 4.8 (ii)]. The last statement can be found in [25, p. 53]. $\rhd$

Theorem 5. *Let X be a Banach lattice and Λ a Dedekind complete AM-space with unit. Then X is a Kaplansky–Hilbert lattice over Λ if and only if the square $X^{\odot}$ is an injective Banach lattice with $\Lambda \simeq \mathscr{Z}_m(X^{\odot})$. In this case the map $\iota : x \mapsto x \odot |x|$ is an isometric order isomorphism from X onto $X^{\odot}$.*

Remark 4. Theorem 5 says that Kaplansky–Hilbert lattices and injective Banach lattices are related as L^2 and L^1. Ivanov [15] proved that if $X := L^2([0,1])$ (and hence $X^{\odot} = L^1([0,1])$), then the bijection $\iota : x \mapsto x \odot |x|$ is also a (non-linear) homeomorphism.

Denote by $\surd$ the inverse of ι, i. e. $\surd(x \odot |x|) = x$ and $\surd(y) \odot |\surd(y)| = y$ for all $x \in X$ and $y \in X^{\odot}$. Then $\|y\|^{\odot} = \|\surd(y)\|^2$ and $\|x\| = \sqrt{\|x \odot |x|\|^{\odot}}$. The maps ι and $\surd$ were named in [15] the *alternating square* and the *alternating square root,* respectively.

A positive operator $T : X \to Y$ (resp. a positive bilinear operator $T : X \times X \to Y$) is said to have the *Levi property* if $\mathrm{im}(T)^{\perp} = \{0\}$ and $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that (Tx_α) (resp. $(T(x_\alpha, x_\alpha))$) is order bounded in Y, see [21, Definition 2.5].

Corollary 1. *A real Banach lattice X is injective if and only if its M-center $\Lambda := \mathscr{Z}_m(X)$ is Dedekind complete and $X^{(2)}$ is a Kaplansky–Hilbert lattice over Λ.*

$\lhd$ By Theorem 5 $X^{(2)}$ is a Kaplansky–Hilbert lattice over Λ if and only if $(X^{(2)})^{\odot}$ is an injective Banach lattice. It remains to observe that $(X^{(2)})^{\odot} = (X^{(2)})^{(1/2)} = X^{(1)} = X$. $\rhd$

Corollary 2. *A Banach lattice X with the Dedekind complete M-center $\Lambda = \mathscr{Z}_m(X)$ is a Kaplansky–Hilbert lattice over Λ if and only if there exists a linear Maharam operator $\Phi : X^{\odot} \to \Lambda$ with the Levi property such that $\|x\| = \sqrt{\|\Phi(x \odot x)\|_\infty}$ $(x \in X)$.*

$\lhd$ This is immediate from Theorem 5 and [21, Theorem 5.1 (3)]. A standard proof (i. e. without involving $\mathrm{V}^{(\mathbb{B})}$) can be given using [6, Theorem 3.1] and [20, Proposition 4.4]. $\rhd$

Corollary 3. *A Banach lattice X is injective if and only if $\Lambda = \mathscr{Z}_m(X)$ is Dedekind complete and there exists a bilinear orthosymmetric Maharam operator $\langle \cdot, \cdot \rangle : X^{(2)} \times X^{(2)} \to \Lambda$ with the Levi property such that $\|x\| = \|\langle \surd(x), \surd(x) \rangle\|_\infty^2$ $(x \in X)$.*

$\lhd$ Apply Corollary 2 with $X := X^{(2)}$, observe that the bilinear operator $\langle x, y \rangle := \Phi(x \odot y)$ is Maharam if and only if so is the linear operator $\Phi : X = (X^{(2)})^{\odot} \to \Lambda$ (see [20, Proposition 4.4]), and take into account [21, Theorem 5.1] with the identity $\langle \surd(x), \surd(x) \rangle = \Phi(|x|)$. $\rhd$

7. Proof of Theorem 5

Everywhere below $\mathscr{X}$ is a Banach lattice in $\mathrm{V}^{(\mathbb{B})}$ and $X = \mathscr{X}{\Downarrow}$.

For a non-empty subset A of a vector lattice, denote by $\vee(A)$ (resp. $\wedge(A)$) the collection of all vectors that can be written as suprema (resp. infima) of finite subsets of A. Put $\wedge\vee(A) := \wedge(\vee(A))$ and $\vee\wedge(A) := \vee(\wedge(A))$. It always turns out that $\vee\wedge(A) = \wedge\vee(A)$. Denote by $\mathscr{P}_{\mathrm{fin}}(A)$ the set of all finite subsets of A.

Lemma 1. *For any nonempty set $A \subset X$ we have $[\![\wedge\vee(A{\uparrow}) = (\wedge\vee(A)){\uparrow}]\!] = \mathbb{1}$.*

$\lhd$ The relation $A \subset \wedge\vee(A)$ implies that $A{\uparrow} \subset \wedge\vee(A){\uparrow}$ and thus $\wedge\vee(A{\uparrow}) \subset \wedge\vee(A){\uparrow}$, since $\wedge\vee(A){\uparrow}$ is a sublattice. For the converse, take $u \in \wedge\vee(A)$ represented as $u = \bigwedge_{k\in n}\bigvee f(k)$ with $n \in \mathbb{N}$ and $f : n := \{0, 1, \ldots, n-1\} \to \mathscr{P}_{\mathrm{fin}}(A)$. Making use of the relation $\mathscr{P}_{\mathrm{fin}}(A{\uparrow}) = \{\theta{\uparrow} : \theta \in \mathscr{P}_{\mathrm{fin}}(A)\}{\uparrow}$, define an internal function $g : n^{\wedge} \to \mathscr{P}_{\mathrm{fin}}(A{\uparrow})$ by $[\![g(k^{\wedge}) = f(k){\uparrow}]\!] = \mathbb{1}$ $(k \in n)$. It is easy to verify that $[\![u = \bigwedge_{k\in n^{\wedge}}\bigvee g(k)]\!] = \mathbb{1}$ and therefore $[\![u \in \wedge\vee(A{\uparrow})]\!] = \mathbb{1}$. Now, if $[\![x \in \wedge\vee(A){\uparrow}]\!] = \mathbb{1}$ then there exists a partition of unity (b_ξ) in $\mathbb{B}$ and a family (u_ξ) in $\wedge\vee(A)$ such that $b_\xi \leqslant [\![x = u_\xi]\!]$ for all ξ. Taking into account that u_ξ is in $\wedge\vee(A{\uparrow})$ we deduce $b_\xi \leqslant [\![x = u_\xi]\!] \wedge [\![u_\xi \in \wedge\vee(A{\uparrow}) \leqslant [\![x \in \wedge\vee(A{\uparrow})]\!]$, whence $[\![x \in \wedge\vee(A{\uparrow})]\!] = \mathbb{1}$. $\rhd$

Denote by $\mathscr{H}(\mathbb{R}^N)$ the vector lattice of all continuous functions $\varphi : \mathbb{R}^N \to \mathbb{R}$ which are *positively homogeneous* ($\equiv \varphi(\lambda t) = \lambda\varphi(t)$ for $\lambda \geqslant 0$ and $t \in \mathbb{R}^N$). If $S := \{t = (t_1, \ldots, t_N) \in \mathbb{R}^N : |t_1| + \cdots + |t_N| = 1\}$, then the map $\varphi \mapsto \varphi|_S$ is a lattice isomorphism from $\mathscr{H}(\mathbb{R}^N)$ onto $C(S)$, the Banach lattice of continuous functions on S. Thus, $\mathscr{H}(\mathbb{R}^N)$ can be also considered as a Banach lattice with the induced norm.

Observe, that $(\mathbb{R}^N)^{\wedge} = (\mathbb{R}^{\wedge})^{N^{\wedge}}$. If $\varphi \in \mathscr{H}(\mathbb{R}^N)$ then $[\![\,\varphi^{\wedge} : (\mathbb{R}^N)^{\wedge} \to \mathbb{R}^{\wedge}$ is a continuous function$\,]\!] = \mathbb{1}$ and $[\![\,$there exists a unique continuous function $\tilde{\varphi} \in \mathscr{H}(\mathscr{R}^{N^{\wedge}})$ such that $\tilde{\varphi}|_{(\mathbb{R}^N)^{\wedge}} = \varphi^{\wedge}\,]\!] = \mathbb{1}$, see [12, Lemma 16]. Evidently, the map $\tau : \varphi \mapsto \tilde{\varphi}$ is a lattice isomorphism from $\mathscr{H}(\mathbb{R}^N)$ into $\mathscr{H}(\mathscr{R}^{N^{\wedge}}){\downarrow}$. Let e_k stands for the kth coordinate function on $\mathbb{R}^N$, i. e. $e_k : (t_1, \ldots, t_N) \mapsto t_k$. Clearly, $\tilde{e_k}$ is a kth coordinate function on $\mathscr{R}^{N^{\wedge}}$.

Lemma 2. *The following holds inside* $\mathrm{V}^{(\mathbb{B})}$*: the Banach lattice $\mathscr{H}(\mathscr{R}^{N^{\wedge}})$ is lattice isomorphic to the completion of the $\mathscr{R}$-normed vector lattice $\mathscr{H}(\mathbb{R}^N)^{\wedge}$ over $\mathbb{R}^{\wedge}$.*

$\lhd$ Recall that if $\mathbb{Q}$ is the field of rationals then $\mathbb{Q}^{\wedge}$ is the field of rationals in $\mathrm{V}^{(\mathbb{B})}$. Denote by $\mathbb{Q}(e_1, \ldots, e_N)$ and $\mathbb{Q}\langle e_1, \ldots, e_N\rangle$ the $\mathbb{Q}$-linear subspace and $\mathbb{Q}$-linear sublattice generated by $\{e_1, \ldots, e_N\}$. Let $\mathbb{Q}^{\wedge}(\tilde{e}_1, \ldots, \tilde{e}_N)$ and $\mathbb{Q}^{\wedge}\langle\tilde{e}_1, \ldots, \tilde{e}_N\rangle$ be the corresponding internal objects in $\mathrm{V}^{(\mathbb{B})}$. If $A := \tau(\mathbb{Q}(e_1, \ldots, e_N))$ and $B := \tau(\mathbb{Q}\langle e_1, \ldots, e_N\rangle)$ then $B = \wedge\vee(A)$ (see [1, Lemma 5.63]) and $B{\uparrow} = \wedge\vee(A{\uparrow})$ by Lemma 1. Moreover, $A{\uparrow} = \mathbb{Q}^{\wedge}(\tilde{e}_1, \ldots, \tilde{e}_N)$, since $\mathbb{Q}(e_1, \ldots, e_N)$ is defined by a restricted formula. The last two observations imply $B{\uparrow} = \mathbb{Q}^{\wedge}\langle\tilde{e}_1, \ldots, \tilde{e}_N\rangle$. It remains to note that $B{\uparrow}$ is uniformly dense in $\mathscr{H}(\mathscr{R}^{N^{\wedge}})$ and is lattice isometric to the sublattice $\mathbb{Q}\langle e_1, \ldots, e_N\rangle^{\wedge}$ which is uniformly dense in $\mathscr{H}(\mathbb{R}^N)^{\wedge}$. $\rhd$

Lemma 3. *Let $x_1, \ldots, x_N \in X$, $\mathbf{x} := (x_1, \ldots, x_N)$, and $\mathfrak{x}$ be an element of* $\mathrm{V}^{(\mathbb{B})}$ *with $[\![\,\mathfrak{x} = (x_1, \ldots, x_{N^{\wedge}})^{\mathbb{B}}\,]\!] = \mathbb{1}$. If $\widehat{\mathbf{x}} : \mathscr{H}(\mathbb{R}^N) \to X$ and $\widehat{\mathfrak{x}} : \mathscr{H}(\mathscr{R}^{N^{\wedge}}) \to \mathscr{X}$ are homogeneous functional calculi in X and $\mathscr{X}$, respectively, then $\widehat{\mathfrak{x}}{\downarrow} \circ \tau = \widehat{\mathbf{x}}$.*

$\lhd$ By Lemma 2 $\widehat{\mathfrak{x}}$ is a unique continuous extension of the map from $\tau\big(\mathscr{H}(\mathbb{R}^N)^{\wedge}\big)$ into $\mathscr{X}$ defined by $\widehat{\mathfrak{x}} : \tilde{\varphi} \mapsto \widehat{\mathbf{x}}(\varphi)$. Therefore, $[\![\,\widehat{\mathfrak{x}}(\tilde{\varphi}) = \widehat{\mathbf{x}}(\varphi)\,]\!] = \mathbb{1}$ for all $\varphi \in \mathscr{H}(\mathbb{R}^N)$. $\rhd$

Lemma 4. *If $0 < \alpha \in \mathbb{R}$, then $(\mathscr{X}^{(\alpha)}){\Downarrow} = (\mathscr{X}{\Downarrow})^{(\alpha)}$. In particular, $\big(\mathscr{X}^{\odot}\big){\Downarrow} = \big(\mathscr{X}{\Downarrow}\big)^{\odot}$.*

$\lhd$ Denote $\mathbf{x} = (x_1, x_2)$, $\mathfrak{x} := (x_1, x_2)^{\mathbb{B}}$ and observe that $(x_1, x_2) \mapsto \widehat{\mathbf{x}}(\sigma_\alpha)$ and $(x_1, x_2) \mapsto \widehat{\mathfrak{x}}(\sigma_\alpha)$ are the operations of addition in X and $\mathscr{X}$, respectively. The addition in $X^{(\alpha)}$ is the bounded descent of the addition in $\mathscr{X}^{(\alpha)}$, since $\widehat{\mathfrak{x}}{\downarrow}(\tilde{\sigma_\alpha}) = \widehat{\mathbf{x}}(\sigma_\alpha)$ by Lemma 3.

Similar assertion about multiplication is evident. $\rhd$

Now we are in a position to prove Theorem 5.

$\vartriangleleft$ Assume that X is a Kaplansky–Hilbert lattice over Λ. By Theorem 2 there is a real Hilbert space $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ such that X and $\mathscr{X}{\Downarrow}$ are unitary equivalent real Kaplansky–Hilbert modules over Λ. In view of [21, Theorem 4.1] $\mathscr{X}$ is also a Banach lattice inside $\mathrm{V}^{(\mathbb{B})}$. Thus, from [27, Corollary 2.7.5] we deduce $[\![\mathscr{X}$ is lattice isometric to $L^2(\mu) := L^2(\Omega, \Sigma, \mu)$ for some measure space $(\Omega, \Sigma, \mu)\,]\!] = \mathbb{1}$. Taking into consideration the relation $(L^2(\mu))^{\odot} = L^1(\mu)$ we conclude that $[\![\mathscr{X}^{\odot}$ is lattice isometric to $L^1(\mu)]\!] = \mathbb{1}$. By Theorem 3 $\mathscr{X}^{\odot}{\Downarrow}$ is an injective Banach lattice with $\Lambda \simeq \mathscr{Z}_m(X^{\odot})$ and it remains to apply Lemma 4.

Conversely, suppose that $X^{\odot}$ is an injective Banach lattice Then, in view of Theorem 3, $X^{\odot}$ is lattice $\mathbb{B}$-isometric to $\mathscr{Y}{\Downarrow}$ for some $\mathscr{Y} \in \mathrm{V}^{(B)}$ with $[\![\,\mathscr{Y} = L^1(\mu) = (L^2(\mu))^{\odot}\,]\!] = \mathbb{1}$. Using again Lemma 4 we deduce $X = (X^{\odot})^{(2)} \simeq_{\mathbb{B}} (\mathscr{Y}{\Downarrow})^{(2)} = ((L^2(\mu))^{\odot}{\Downarrow})^{(2)} \simeq_{\mathbb{B}} L^2(\mu){\Downarrow}$. By Theorem 2 and [21, Theorem 4.1] X is a Kaplansky–Hilbert lattice over Λ. $\vartriangleright$

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ПРИНЦИП КАНТОРОВИЧА В ДЕЙСТВИИ: AW^*-МОДУЛИ И ИНЪЕКТИВНЫЕ БАНАХОВЫ РЕШЕТКИ

Кусраев А. Г.

Используя методы булевозначного анализа установлено, что решетки Капланского — Гильберта и инъективные банаховы решетки могут быть преобразованы друг в друга при помощи процедуры овыпукления. Обсуждается также взаимосвязь между эвристическим принципом переноса Канторовича и принципом переноса в булевозначном анализе.

Ключевые слова: принцип Канторовича, векторная решетка, булевозначный анализ, булевозначное представление, модуль Капланского — Гильберта, инъективная банахова решетка, оператор Магарам, квадрат векторной решетки, овыпукление.