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CN-EDGE DOMINATION IN GRAPHS

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Let $G = (V, E)$ be a graph. A subset D of V is called common neighbourhood dominating set (CN-dominating set) if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\Gamma(u, v)| \geqslant 1$, where $|\Gamma(u, v)|$ is the number of common neighbourhood between the vertices u and v. The minimum cardinality of such CN-dominating set denoted by $\gamma_{cn}(G)$ and is called common neighbourhood domination number (CN-edge domination) of G. In this paper we introduce the concept of common neighbourhood edge domination (CN-edge domination) and common neighbourhood edge domatic number (CN-edge domatic number) in a graph, exact values for some standard graphs, bounds and some interesting results are established.

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1. Introduction

By a *graph* $G = (V, E)$ we mean a finite and undirected graph with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. A set D of vertices in a graph G is a *dominating* set if every vertex in $V - D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A *line graph* $L(G)$ (also called an interchange graph or edge graph) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. For terminology and notations not specifically defined here we refer reader to [5]. For more details about domination number and its related parameters, we refer to [6], [9], and [10].

Let G be a simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $i \neq j$, the *common neighborhood* of the the vertices v_i and v_j, denoted by $\Gamma(v_i, v_j)$, is the set of vertices, different from v_i and v_j, which are adjacent to both v_i and v_j. A subset D of V is called common neighbourhood dominating set (CN-*dominating set*) if for every $v \in V - D$ there exist a vertex $u \in D$ such that $uv \in E(G)$ and $|\Gamma(u, v)| \geqslant 1$, where $|\Gamma(u, v)|$ is the number of common neighbourhood between the vertices u and v. The minimum cardinality of such CN-dominating set denoted by $\gamma_{cn}(G)$ and is called *common neighbourhood domination number* (CN-domination number) of G. The CN-domination number is defined for any graph. A common neighbourhood dominating set D is said to be *minimal* if no proper subset of D is common neighbourhood dominating set. If $u \in V$, then the CN-neighbourhood of u denoted by $N_{cn}(u)$ is defined as $N_{cn}(u) = \{v \in N(u) : |\Gamma(u, v)| \geqslant 1\}$. The cardinality of $N_{cn}(u)$ is denoted by $\deg_{cn}(u)$ in G, and $N_{cn}[u] = N_{cn}(u) \cup \{u\}$. The maximum and minimum common neighbourhood degree of a vertex in G are denoted respectively by $\Delta_{cn}(G)$ and $\delta_{cn}(G)$. That is $\Delta_{cn}(G) = \max_{u \in V} |N_{cn}(u)|$, $\delta_{cn}(G) = \min_{u \in V} |N_{cn}(u)|$. A subset S of V is called a *common neighbourhood independent set* (CN-independent set), if for every $u \in S, v \notin N_{cn}(u)$ for all $v \in S - \{u\}$. It is clear that every independent set is CN-independent set. An CN-independent set S is called *maximal* if any vertex set properly containing S is not CN-independent set. The maximum cardinality of CN-independent set is denoted by β_{cn}, and the lower CN-independence number i_{cn} is the minimum cardinality of the CN-maximal independent set. An edge $e = uv \in E(G)$ is said to be *common neighbourhood edge* (CN-*edge*) if $|\Gamma(u, v)| \geqslant 1$. For more details about CN-dominating set see [1]. The concept of edge domination was introduced by Mitchell and Hedetniemi [8]. Let $G = (V, E)$ be a graph. A subset X of E is called an edge dominating set of G if every edge in $E - X$ is adjacent to some edge in X.

In this paper analogue to the edge domination, we introduce the concept of common neighbourhood edge domination (CN-edge domination) in a graph and common neighbourhood edge domatic number (CN-edge domatic number) in a graph, exact values for the some standard graphs bounds and some interesting results are established.

2. CN-Edge Domination Number

Let $G = (V, E)$ be a graph and f, e be any two edges in E. Then f and e are adjacent if they have one end vertex in common.

DEFINITION 2.1. Two edges f and e are *common neighbourhood adjacent* (CN-*adjacent*) if f adjacent to e and there exist another edge g adjacent to both f and e.

DEFINITION 2.2. A set S of edges is called *common neighbourhood edge dominating set* (CN-*edge dominating set*) if every edge f not in S is CN-adjacent to at least one edge $f' \in S$. The minimum cardinality of such CN-edge dominating set is denoted by $\gamma'_{cn}(G)$ and called CN-edge domination number of G.

The CN-edge neighbourhood of f denoted by $N_{cn}(f)$ is defined as $N_{cn}(f) = \{g \in E(G) : f \text{ and } g \text{ are CN-adjacent}\}$. The cardinality of $N_{cn}(f)$ is called the CN-*degree* of the edge f and denoted by $\deg_{cn}(f)$. The maximum and minimum CN-degree of edges in G are denoted respectively by $\Delta'_{cn}(G)$ and $\delta'_{cn}(G)$. That is $\Delta'_{cn}(G) = \max_{f \in E(G)} |N_{cn}(f)|$, $\delta'_{cn}(G) = \min_{f \in E(G)} |N_{cn}(f)|$.

A common neighbourhood edge dominating set S is minimal if for any edge $f \in S$, $S - \{f\}$ is not CN-edge dominating set of G. A subset S of E is called *CN-edge independent set*, if for any $f \in S$, $f \notin N_{cn}(g)$, for all $g \in S - \{f\}$. If an edge $f \in E$ be such that $N_{cn}(f) = \phi$ then j is in any CN-dominating set. Such edges are called CN-*isolated*. The minimum CN-edge dominating set denoted by γ'_{cn}-set.

An edge dominating set X is called an CN-*independent edge dominating set* if no two edges in X are CN-adjacent. The CN-independent edge domination number $\gamma'_{cni}(G)$ is the minimum cardinality taken over all CN-independent edge dominating sets of G. The CN-edge independence number $\beta'_{cn}(G)$ is defined to be the number of edges in a maximum CN-independent set of edges of G. For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

In Figure 1, $E(G) = \{a, b, c, d, e\}$. The minimal edge dominating sets are $\{b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$, $\{b, c\}$, $\{b, d\}$, $\{b, e\}$. Therefore $\gamma'(G) = 1$. The minimal CN-edge dominating sets are $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$. Therefore $\gamma'_{cn}(G) = 2$. The edge a is CN-edge isolated but not edge isolated.

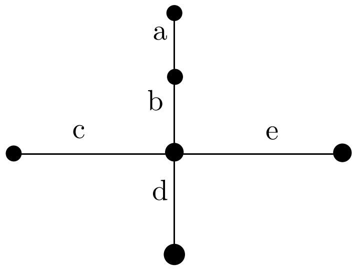

Fig. 1.

Observation 2.1. *For any graph G with at least one edge, $1 \leqslant \gamma'_{cn}(G) \leqslant q$.*

If the graph G is triangle free and claw free and $\Delta(G) \leqslant 2$, then for any two adjacent edge e and f there is no any edge adjacent both e and f, so we have the following proposition.

Proposition 2.1. *Let $G = (V, E)$ be a nontrivial graph with q edges. Then $\gamma'_{cn}(G) = q$ if and only G triangle free graph with $\Delta(G) \leqslant 2$.*

Hence it follows that

$$\gamma'_{cn}(C_p) = \gamma_{cn}(C_p) = p \text{ and } \gamma'_{cn}(P_p) = \gamma_{cn}(P_{p-1}) = p - 1.$$

From the definition of line graph and the CN-edge domination the following Proposition is immediate.

Observation 2.2. *For any graph G, we have $\gamma'_{cn}(G) = \gamma_{cn}(L(G))$.*

Proposition 2.2. *For any complete graph K_p and Complete bipartite graph $K_{n,m}$, we have*

$$\gamma'_{cn}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor \quad \text{and} \quad \gamma'_{cn}(K_{n,m}) \min\{m, n\}.$$

Proposition 2.3. *For any wheel graph W_p of p vertices, we have*

$$\gamma'_{cn}(W_p) = \begin{cases} 1 + \frac{p-3}{3}, & \text{if } p \cong 0 (\operatorname{mod} 3); \\ 1 + \frac{p-1}{3}, & \text{if } p \cong 1 (\operatorname{mod} 3); \\ 1 + \frac{p-2}{3}, & \text{if } p \cong 2 (\operatorname{mod} 3). \end{cases}$$

Obviously for any graph G any CN-edge dominating set is edge dominating set then the following proposition follows.

Proposition 2.4. *For any graph G, $\gamma'_{cn}(G) \geqslant \gamma'(G)$.*

Theorem 2.1. *The CN-edge dominating set F is minimal if and only if for each edge $f \in F$ one of the following conditions holds*

(*i*) *$N_{cn}(f) \cap F = \phi$;*

(*ii*) *there exist an edge $g \in E - F$ such that $N_{cn}(g) \cap F = \{f\}$.*

$\lhd$ Suppose that F is a minimal CN-edge dominating set. Assume that (i) and (ii) do not hold. Then for some $f \in F$ there exist an edge $g \in N_{cn}(f) \cap F$ and for every edge $h \in E - F$, $N_{cn}(h) \cap F \neq \{f\}$. Therefore $F - \{f\}$ is CN-edge dominating set contradiction to the minimality of F. Therefor (i) or (ii) holds.

Conversely, suppose for every $f \in F$ one of the conditions holds. Suppose F is not minimal. Then there exist $f \in F$ such that $F - \{f\}$ is CN-edge dominating set. Therefore there exist an edge $g \in F - \{f\}$ such that $g \in N_{cn}(f)$. Hence f does not satisfy (i). Then f

must satisfy (ii). Then there exist an edge $g \in E - F$ such that $N_{cn}(g) \cap F = \{f\}$. Since $F - \{f\}$ is CN-edge dominating set there exist an edge $f' \in F - \{f\}$ such that f' is CN-adjacent to g. Therefore $f' \in N_{cn}(g) \cap F$ and $f' \neq f$, a contradiction to $N_{cn}(g) \cap F = \{f\}$. Hence F is minimal CN-edge dominating set. $\triangleright$

Proposition 2.5. *For any Graph G without any CN-isolated edges, if F is minimal CN-edge dominating set then $E - F$ is CN-edge dominating set.*

$\triangleleft$ Let F be minimal CN-edge dominating set of G. Suppose $E - F$ is not CN-edge dominating set. Then there exist an edge f such that $f \in F$ is not CN-adjacent to any edge in $E - F$. Since G has no CN-isolated edges then f is CN-dominated by at least one edge in $F - \{f\}$. Thus $F - \{f\}$ is CN-edge dominating set a contradiction to the minimality of F. Therefore $E - F$ is is CN-edge dominating set. $\triangleright$

Proposition 2.6. *For any graph G, any CN-independent edge set F is maximal CN-independent edge set if and only if it is CN-edge independent and CN-edge dominating set of G.*

$\triangleleft$ Suppose F be maximal CN-independent set of G. Then for every edge $f \in E - F$, the set $F \cup \{f\}$ is not CN-independent, that is for every edge $f \in E - F$, there is an edge $g \in F$ in such that f is CN-adjacent to g. Thus F is CN-edge dominating set. Hence F is both CN-edge independent and CN-edge dominating set of G.

Conversely, suppose F is both CN-edge independent and CN-edge dominating set of G. Suppose F is not maximal CN-edge independent set. Then there exist an edge $f \in E - F$ such that $F \cup \{f\}$ is CN-independent, then there is no edge in F is CN-adjacent to f. Hence F is not CN-edge dominating set which is a contradiction. Hence F is maximal CN-independent set. $\triangleright$

Theorem 2.2. *For any graph G, $\gamma'_{cn}(G) \leqslant q - \Delta'_{cn}(G)$.*

$\triangleleft$ Let f be an edge in G such that $\deg_{cn}(f) = \Delta'_{cn}(G)$. Then $E(G) - N_{cn}(f)$ is CN-edge dominating set. Hence $\gamma'_{cn}(G) \leqslant q - \Delta'_{cn}(G)$. $\triangleright$

Theorem 2.3. *For any γ'_{cn}-set F of a graph $G = (V, E)$, $|E - F| \leqslant \sum_{f \in F} \deg_{cn}(f)$ and the equality holds if and only if*

(i) F is CN-independent edge

(ii) for every edge $f \in E - F$, there exists only one edge $g \in F$ such that $N_{cn}(f) \cap F = \{g\}$.

$\triangleleft$ Since each edge in $E - F$ is CN-adjacent to at least one edge of F. Therefore each edge in $E - F$ contributes at least one to the sum of the CN-degrees of the edges of F. Hence

$$|E - F| \leqslant \sum_{f \in F} \deg_{cn}(f).$$

Let $|E - F| = \sum_{f \in F} \deg_{cn}(f)$ and suppose that F is not CN-independent edge. Clearly each edge in $E - F$ is counted in the sum $\sum_{f \in F} \deg_{cn}(f)$. Hence if f_1 and f_2 are CN-adjacent edges, then f_1 is counted in $\deg_{cn}(f_1)$ and vice versa. Then the sum exceeds $|E - F|$ by at least two, contrary to the hypothesis. Hence F must be CN-independent edge.

Now suppose (ii) is not true. Then $N_{cn}(f) \cap F \geqslant 2$ for $f \in E - F$. Let f_1 and f_2 belong to $N_{cn}(f) \cap F$, hence $\sum_{f \in F} \deg_{cn}(f)$ exceed $E - F$ by at least one since f counted twice, once in $\deg_{cn}(f_1)$ and the other in $\deg_{cn}(f_2)$. Hence if the equality holds then the condition (i) and (ii) must be true. The converse is obvious. $\triangleright$

Theorem 2.4. *For any (p, q) graph G, $\left\lceil \frac{q}{\Delta'_{cn}(G)+1} \right\rceil \leqslant \gamma'_{cn}(G) \leqslant q - \beta'_{cn} + q_0$, where q_0 is the number of CN-isolated edges.*

◁ From the previous proposition $|E - F| \leqslant \sum_{f \in F} \deg_{cn}(f) \leqslant \gamma'_{cn}(G)\Delta'_{cn}$.
Hence $q - \gamma'_{cn}(G) \leqslant \gamma'_{cn}(G)\Delta'_{cn}$. Therefore

$$\left\lceil \frac{q}{\Delta'_{cn}(G) + 1} \right\rceil \leqslant \gamma'_{cn}(G). \tag{1}$$

Let $G' = \langle E(G) - I_{cn}(G) \rangle$ where $I_{cn}(G)$ is the set of CN-isolated edges of G. Let S be the maximal CN-independent set of edges of G'. Hence S is also CN-edge dominating set of G'. Since G' does not have CN-isolated edges $E(G') - S$ is also CN-edge dominating set of G'. Therefore $\gamma'_{cn}(G') \leqslant |E(G') - S| = q(G') - \beta'_{cn}(G')$. But $\gamma'_{cn}(G') = \gamma'_{cn}(G) - q_0$ and $q(G') = q(G) - q_0$ and $\beta'_{cn}(G') = \beta'_{cn}(G) - q_0$. Hence

$$\gamma'_{cn}(G) - q_0 \leqslant q(G) - q_0 - \big(\beta'_{cn}(G) - q_0\big).$$

Therefore

$$\gamma'_{cn}(G) \leqslant q - \beta'_{cn} + q_0. \tag{2}$$

From (1) and (2) we have

$$\left\lceil \frac{q}{\Delta'_{cn}(G) + 1} \right\rceil \leqslant \gamma'_{cn}(G) \leqslant q - \beta'_{cn} + q_0. \ \triangleright$$

3. CN-Edge Domatic Number

Definition 3.1. The maximum order of partition of the edges $E(G)$ into CN-edge dominating sets is called CN-*edge domatic number* of G and denoted by $d'_{cn}(G)$.

Observation 3.1. *For any graph G, $d'_{cn}(G) \leqslant d'(G)$, where $d'(G)$ is the edge domatic number.*

◁ Let $G = (V, E)$ be a graph Since any partition of $E(G)$ into CN-edge domination set is also partition of $E(G)$ into edge dominating set. Hence $d'_{cn}(G) \leqslant d'(G)$. ▷

Proposition 3.1. (*i*) *For any cycle C_p and path P_p with p vertices $d'_{cn}(C_p) = d'_{cn}(P_p) = 1$.*
(*ii*) *For any complete bipartite graph $K_{m,n}$, $d'_{cn}(K_{m,n}) = \max\{m, n\}$.*
(*iii*) *$d'_{cn}(G) = 1$ if and only if G has at least one CN-isolated edge.*

Proposition 3.2. *Let G be a complete graph K_p with $p \geqslant 2$ vertices. Then*

$$d'_{cn}(G) = \begin{cases} p - 1, & \text{if } n \text{ is even;} \\ p, & \text{if } n \text{ is odd.} \end{cases}$$

◁ If p is even, then K_p can be decomposed into $p - 1$ pairwise edge-disjoint linear factors. The edge set of each these factors is an edge dominating set in K_p. Since any two adjacent vertices in the complete graph are also CN-adjacent, then any edge dominating set is also CN-edge dominating set in K_p. Hence $d'_{cn}(K_p) \geqslant p - 1$. Suppose that $d'_{cn}(K_p) \geqslant p$ and consider ζ is an CN-edge domatic partition of K_p with p classes the mean value of the orders of these classes has at most $\frac{p-1}{2}$. This implies that at least one of the classes has at most $\left[\frac{p-1}{2}\right] = \frac{p}{2} - 1$ edges. But the partition covers at most $p - 2$ vertices, there are two vertices no edge in ζ incident to any of the two vertices. Hence there is no edge in ζ CN-adjacent to the edge joining these vertices. Therefore $d'_{cn}(K_p) = p - 1$ if p is even.

Now let p is odd. By labeling the vertices of K_p by $x_1, x_2, \ldots, x_p$. We can make the following domatic partition of the vertices $E_1, \ldots, E_p$, where $E_i = x_{i+j}x_{i-j+1}$, where $j = 1, \ldots, \frac{p-1}{2}$ and the scripts will be taken modulo p. Hence $d'_{cn}(K_p) \geqslant p$. If we suppose that $d'_{cn}(K_p) \geqslant p+1$, then in the same way of the first case we can prove that there exist an CN-edge domatic partition one of whose classes has at most $\frac{p-3}{2}$ edges this set cover at most $p-3$ vertices and it is not an CN-edge dominating set a contradiction. Hence $d'_{cn}(K_p) = p$ if p is odd. $\rhd$

Theorem 3.1. *For any graph G with q edges, $d'_{cn}(G) \leqslant \frac{q}{\delta'_{cn}(G)}$.*

$\lhd$ Assume that $d'_{cn}(G) = d$ and $\{D_1, D_2, \ldots, D_d\}$ is a partition of $E(G)$ into d CN-edge dominating sets, clearly $|D_i| \geqslant \gamma'_{cn}(G)$ for $i = 1, 2, \ldots, d$ and we have $q = \sum_{i=1}^{d} |D_i| \geqslant d\delta'_{cn}(G)$.

Hence $d'_{cn}(G) \leqslant \frac{q}{\delta'_{cn}(G)}$. $\rhd$

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CN-РЕБЕРНОЕ ДОМИНИРОВАНИЕ В ГРАФАХ

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Пусть $G = (V, E)$ — граф. Подмножество D множества V называется реберно доминирующим множеством с общей окрестностью (CN-реберно доминирующим множеством), если для любой вершины $v \in V - D$ существует вершина $u \in D$ такая, что $uv \in E(G)$ и $|\Gamma(u, v)| \geqslant 1$, где $|\Gamma(u, v)|$ — множество общих соседей вершин u и v. Наименьшая мощность такого CN-реберно доминирующего множества обозначается $\gamma_{cn}(G)$ и называется реберно доминирующим числом с общей окрестностью (CN-реберно доминирующим числом) графа G. В данной статье вводятся понятия реберно доминирующего числа с общей окрестностью и реберно доматического числа с общей окрестностью (CN-реберно доматического числа) в графе, найдены их точные значения в некоторых стандартных графах, установлены границы и некоторые интересные результаты.

Ключевые слова: реберно доминирующее множество с общей окрестностью, реберно доматическое число с общей окрестностью, реберно доминирующее число с общей окрестностью.