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SOLUTIONS OF THE DIFFERENTIAL INEQUALITY WITH A NULL LAGRANGIAN: HIGHER INTEGRABILITY AND REMOVABILITY OF SINGULARITIES. I[1]

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Dedicated to Academician Yuri Grigor'evich Reshetnyak
on the occasion of his 85th birthday

The aim of this paper is to derive the self-improving property of integrability for derivatives of solutions of the differential inequality with a null Lagrangian. More precisely, we prove that the solution of the Sobolev class with some Sobolev exponent slightly smaller than the natural one determined by the structural assumption on the involved null Lagrangian actually belongs to the Sobolev class with some Sobolev exponent slightly larger than this natural exponent. We also apply this property to improve Hölder regularity and stability theorems of [19].

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Introduction

In this paper and [22] we establish higher integrability and removability properties of solutions $v\colon V \to \mathbb{R}^m$, $V \subset \mathbb{R}^n$, of the following inequality

$$F(v'(x)) \leqslant KG(v'(x)) + H(x) \quad \text{a.e.} \quad V \tag{1}$$

constructed by means of a continuous function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, a null Lagrangian $G : \mathbb{R}^{m\times n} \to \mathbb{R}$, a measurable function $H\colon V \to \mathbb{R}$, and a constant $K \geqslant 1$. Here $v'(x)$ denotes the differential of v at $x \in V$. In the case $H(x) \equiv 0$ this inequality has the form

$$F(v'(x)) \leqslant KG(v'(x)) \quad \text{a.e.} \quad V. \tag{2}$$

In [19], the author has obtained some results on closure of sets of solutions to (2) with respect to the local convergence in the Lebesgue space and their Hölder regularity (for example, see

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[19, Theorems 7 and 8]). Using these results, the author has proved the theorems ([19, Theorems 1 and 3–6]) on stability of the class of solutions to the equation

$$F(u'(x)) = G(u'(x)) \quad \text{a. e. } \ x \in V \tag{3}$$

(also see [16–18]).

Our main results are analogs of the well-known higher integrability and removability theorems for mappings with bounded distortion (quasiregular mappings). A mapping $v \in W^{1,n}_{\mathrm{loc}}(V;\mathbb{R}^n)$ of an open set $V \subset \mathbb{R}^n$ is an (sense-preserving) mapping with K-bounded distortion, $K \geqslant 1$, if v satisfies the distortion inequality

$$|v'(x)|^n \leqslant K \det v'(x) \quad \text{a. e. } V, \tag{4}$$

where $|v'(x)|$ is the operator norm of the matrix $v'(x)$. If, in addition, v is topological, then v is K-quasiconformal. The distortion inequality is the particular case of (2) with the following functions $F(v'(x)) = |v'(x)|^n$ and $G(v'(x)) = \det v'(x)$. The theory of quasiconformal mappings and mappings with bounded distortion is the key part of modern geometric analysis which has many diverse applications (for example, see monographs [2, 6, 10, 30, 31, 32, 36, 41, 45, 46, 49, 52, 53, 58, 59, 60, 61, 62, 63, 68, 69] and the bibliography therein).

The higher integrability for planar mappings with bounded distortion was established by B. Bojarski [8, 9]. More precisely, he proved that there exists an exponent $\overline{p}(2, K) > 2$ such that mappings with K-bounded distortion (*a priori* in $W^{1,2}_{\mathrm{loc}}$) belong to $W^{1,s}_{\mathrm{loc}}$ for every $s < \overline{p}(2, K)$. F. W. Gehring [28] has extended this result to n-dimensional quasiconformal mappings. The existence of an exponent $\overline{p}(n, K) > n \geqslant 3$ such that all n-dimensional mappings with K-bounded distortion lie in $W^{1,s}_{\mathrm{loc}}$ for every $s < \overline{p}(n, K)$ was obtained by A. Elcart and N. G. Meyers [54] and, independently, by Yu. G. Reshetnyak [57] (also see [29, 50, 51, 58, 59, 60, 61, 62]). Moreover, Yu. G. Reshetnyak [57] has established that $\overline{p}(n, K) \to \infty$ as $K \to 1$ (also see [33]). For $n = 2$ this result was obtained by O. Lehto [48]. The higher integrability result has a dual version. In two papers, T. Iwaniec and G. Martin [35] (for even dimensions) and T. Iwaniec [34] (for all dimensions) have proved that there exists an exponent $1 < \underline{p}(n, K) < n$ such that if a mapping $v \in W^{1,p}_{\mathrm{loc}}$ with some $p > \underline{p}(n, K)$ satisfies inequality (4), i. e. v is a weakly quasiregular mapping, then v belongs to $W^{1,s}_{\mathrm{loc}}$ for every $s < \overline{p}(n, K)$ (also see [23, 36, 37] and for $n = 2$ the monograph [49]). Here the word “weakly” means that the Sobolev integrable exponent p of v may be smaller then the dimension n. In this case, $\det v'(x)$ need not be locally integrable. Thus the natural exponent for the distortion inequality is the dimension n. K. Astala [1] proved that $\underline{p}(2, K) = 2K/(K+1)$ and $\overline{p}(2, K) = 2K/(K-1)$ are the sharp exponents for higher integrability of planar mappings with bounded distortion (also see [2]).

Also, higher integrability results have been established for the following mapping classes: the classes of mappings that are close to multidemensional holomorphic mappings [39, 40, 41]; the classes of mappings that are close to solutions of linear elliptic partial differential equations [7, 14, 41]; the classes of quasihomoteties [64, 65]; the classes of quasiregular mappings of several n-dimensional variables [12, 13, 14]; the classes of weakly (K_1, K_2)-quasiregular mappings [24, 27, 67]; the classes of degenerate weakly (K_1, K_2)-quasiregular mappings [26]; the classes of weakly $(K_1, K_2(x))$-quasiregular mappings of several n-dimensional variables [25]; and a series of other calsses [41–43]. Mappings of these classes, as mappings with bounded distortion, can be considered as solutions to (1) with specific functions F, G, and H (some examples of such considerations can be found in [19, § 2]). Our main higher integrability result (Theorem 2.1) contain (either partially, or fully, or in an improved form) some of the known results on higher integrability for mappings of these classes. We also apply this result to improve the above-mentioned theorems on Hölder regularity and stability. In this paper, as in [16–19], we develop approaches and methods used for investigations of mappings with bounded distortion to study properties of solutions to (1). Some results of this paper have been announced in [20, 21].

The main aim of this paper is to prove Theorem 2.1 on higher integrability of solutions to (1). Applying this theorem, in the next paper [22], we establish a result on removability of singularities for solutions to (1). Also, in [22], using the Hodge decomposition theory developed by T. Iwaniec and G. Martin [34–36], we derive integral estimates for minors of Jacobian matrixs. These estimates have independent interest. In this paper they are used in the proof of Theorem 2.1.

This paper is organized as follows. In § 1 we give the basic notation and terms. The main results are stated in § 2. In § 3 we expose some auxiliary lemmas. The proof of the higher integrability theorem (Theorem 2.1 is given in § 4).

1. Notation and Terminology

Let A be a set in $\mathbb{R}^n$. The topological boundary of A is denoted by ∂A. The diameter of A is defined as $\operatorname{diam} A := \sup\{|x-y| : x, y \in A\}$. The outer Lebesgue measure of A is denoted by $|A|$. We use the symbol $\dim_H A$ for the Hausdorff dimension of A.

The set $\mathbb{R}^{m\times n} := \Big\{\zeta = (\zeta_{\mu\nu})_{\substack{\mu=1,\dots,m\\ \nu=1,\dots,n}} : \zeta_{\mu\nu} \in \mathbb{R},\ \mu = 1,\dots,m,\ \nu = 1,\dots,n\Big\}$ consists of all real $m \times n$-matrices. We identify a matrix $\zeta = (\zeta_{\mu\nu})_{\substack{\mu=1,\dots,m\\ \nu=1,\dots,n}} \in \mathbb{R}^{m\times n}$ with the linear mapping $(\zeta_1,\dots,\zeta_m)\colon \mathbb{R}^n \to \mathbb{R}^m$, where $\zeta_\mu(x) := \sum_{\nu=1}^n \zeta_{\mu\nu} x_\nu$, $\mu = 1,\dots,m$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. The operator norm in $\mathbb{R}^{m\times n}$ is defined as $|\zeta| := \sup\{|\zeta(x)| : x \in \mathbb{R}^n,\ |x| < 1\}$. The number of k-tuples of ordered indices in $\Gamma_n^k := \{I = (i_1,\dots,i_k) : 1 \leqslant i_1 < \dots < i_k \leqslant n,\ i_\varkappa \in \{1,\dots,n\},\ \varkappa = 1,\dots,k\}$ equals the binomial coefficient $\binom{n}{k} := \frac{n!}{k!(n-k)!}$. Given $x \in \mathbb{R}^n$ and $I \in \Gamma_n^k$, we put $x_I := (x_{i_1},\dots,x_{i_k}) \in \mathbb{R}^k$. If $I \in \Gamma_n^k$ and $J \in \Gamma_m^k$, then $\det_{JI}\zeta := \det\begin{pmatrix} \zeta_{j_1 i_1} & \cdots & \zeta_{j_1 i_k} \\ \vdots & \ddots & \vdots \\ \zeta_{j_k i_1} & \cdots & \zeta_{j_k i_k}\end{pmatrix}$ is the $k\times k$-minors of the matrix $\zeta \in \mathbb{R}^{m\times n}$. For $\varepsilon > -k$ we put $|\zeta|^\varepsilon \det_{JI}\zeta = 0$ at $\zeta = 0$.

The Jacobian matrix of $u = (u_1,\dots,u_m)\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ at a point $x \in U$ is the matrix $u'(x) := \big(\frac{\partial u_\mu}{\partial x_\nu}(x)\big)_{\substack{\mu=1,\dots,m\\ \nu=1,\dots,n}}$. If $I \in \Gamma_n^k$ and $J \in \Gamma_m^k$ then $\frac{\partial u_J}{\partial x_I}(x) = \frac{\partial(u_{j_1},\dots,u_{j_k})}{\partial(x_{i_1},\dots,x_{i_k})}(x) := \det_{JI} u'(x)$ and $\partial_I u_\mu(x) := \Big(\frac{\partial u_\mu}{\partial x_{i_1}}(x),\dots,\frac{\partial u_\mu}{\partial x_{i_1}}(x)\Big)$, $\mu = 1,\dots,m$. For $h\colon U \to \mathbb{R}$ we put $h_+(x) := \sup(h(x), 0)$, $x \in U$.

Let $\mathscr{V}$ be a real vector space equipped with a norm $|\cdot|$. We say that a function $\Phi\colon \mathscr{V} \to \mathbb{R}$ is *positively homogeneous of degree* $p \in \mathbb{R}$ if $\Phi(tx) = t^p\Phi(x)$ for all $t > 0$ and $x \in \mathscr{V} \setminus \{0\}$. For $\varepsilon > -1$ we put $|x|^\varepsilon x = 0$ at $x = 0$.

Following Ch. B. Morrey [55], we say that a continuous function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is *quasiconvex*, if

$$|B(0,1)|F(\zeta) \leqslant \int\limits_{B(0,1)} F(\zeta + \varphi'(x))\,dx \tag{5}$$

for all $\varphi \in C_0^\infty(B(0,1);\mathbb{R}^m)$ and $\zeta \in \mathbb{R}^{m\times n}$. Let $p \geqslant 1$. Following M. A. Sychev [66], we say that a quasiconvex function F is *strictly p-quasiconvex* if, for $\zeta \in \mathbb{R}^{m\times n}$ and $\varepsilon, C > 0$, there is $\delta = \delta(\zeta, \varepsilon, C) > 0$ such that, for each mapping $\varphi \in C_0^\infty(B(0,1);\mathbb{R}^m)$ satisfying $\|\varphi'\|_{L^p(B(0,1);\mathbb{R}^{m\times n})} \leqslant C|B(0,1)|^{1/p}$, the condition $\int_{B(0,1)} F(\zeta+\varphi'(x))\,dx \leqslant |B(0,1)|(F(\zeta)+\delta)$ implies $|\{x \in B(0,1) : |\varphi'(x)| \geqslant \varepsilon\}| \leqslant \varepsilon|B(0,1)|$. Observe that in the mathematical literature the term strictly quasiconvexity is also used for another property (which is close but nonequivalent to ours) consisting in the fact that the strict inequality in the definition of quasiconvexity (5) is valid for nonzero mappings φ (for example, see [38]). In this article we use the term in the sense of M. A. Sychev's definition [66]. In the case $p > 1$ the notion of strictly p-quasiconvexity for functions F of this article is equivalent to the notion of strictly closed p-quasiconvexity from J. Kristensen's article [44] which is defined in terms of the theory of gradient Young measures (see [44, Proposition 3.4]). Observe that we can replace the ball $B(0,1)$ in the definitions of quasiconvexity and strictly p-quasiconvexity by an arbitrary bounded domain U with $|\partial U| = 0$ (for example, see [56]). A function $G\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is a *null Lagrangian* if both functions G and $-G$ are quasiconvex. The term "null Lagrangian" appeared due to the following fact: The Euler–Lagrange equation corresponding to the variational integral $\int_U G(u'(x))\,dx$ with null Lagrangian G holds identically for all admissible mappings $u\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ (see [4] and also [5, 36, 56]). The only the affine combinations of minors (called *quasiaffine functions*) are null Lagrangians [15, 47] (also see [3, 4, 5, 36, 55, 56]); i. e.

$$G(\zeta) = \gamma_0 + \sum_{k=1}^{\min\{m,n\}} \sum_{J\in\Gamma_m^k, I\in\Gamma_n^k} \gamma_{JI} \det_{JI} \zeta, \quad \zeta \in \mathbb{R}^{m\times n}, \tag{6}$$

for some $\gamma_0, \gamma_{JI} \in \mathbb{R}$.

2. Statement of the Main Results

The principal result on higher integrability of solutions to (1) states as follows

Theorem 2.1 (Higher integrability). *Let $n, m, k \in \mathbb{N}$ and $t > k$ such that $2 \leqslant k \leqslant \min\{n, m\}$, and let V be an open set in $\mathbb{R}^n$. Suppose that a continuous function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfies*

$$F(\zeta) \geqslant c_F|\zeta|^k, \quad \zeta \in \mathbb{R}^{m\times n}, \tag{7}$$

with some constant $c_F > 0$, a null Lagrangian $G\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is homogeneous of degree k, and a measurable function $H\colon V \to \mathbb{R}$ has $H_+ \in L^t_{\mathrm{loc}}(V)$. Then for $K \geqslant 1$ there exist two numbers $\underline{p} = \underline{p}(F, G, K)$ and $\overline{p} = \overline{p}(F, G, K)$ depended only F, G, and K with $1 \leqslant \underline{p} < k < \overline{p} \leqslant t$ such that for a given exponent $p > \underline{p}$ every solution $v \in W^{1,p}_{\mathrm{loc}}(V;\mathbb{R}^m)$ to (1) actually lies in $W^{1,s}_{\mathrm{loc}}(V;\mathbb{R}^m)$ for all $s \in (\underline{p}, \overline{p})$. Moreover, for a number $\varepsilon > 0$, a vector $b \in \mathbb{R}^m$, and a test function $\varphi \in C_0^\infty(V)$ we have the Caccioppoli-type inequality

$$\|\varphi(\cdot)v'(\cdot)\|_{L^s(V;\mathbb{R}^{m\times n})} \leqslant C\||(v(\cdot)-b)\otimes\varphi'(\cdot)| + |\varphi(\cdot)|(\varepsilon + \varepsilon^{1-k}H_+(\cdot))\|_{L^s(V)} \tag{8}$$

with some constant $C = C(F, G, K, s) > 0$ depended only F, G, K, and s.

In the case $H(x) \equiv 0$ a straightforward consequence of Theorem 2.1 is the following.

Corollary 2.2. *Under the conditions of Theorem 2.1, there exist two numbers $\underline{p}_0 = \underline{p}_0(F, G, K)$ and $\overline{p}_0 = \overline{p}_0(F, G, K)$ depended only F, G, and K with $1 \leqslant \underline{p}_0 < k < \overline{p}_0$ such that for a given exponent $p > \underline{p}_0$ every solution $v \in W^{1,p}_{\mathrm{loc}}(V;\mathbb{R}^m)$ to (2) actually lies in $W^{1,s}_{\mathrm{loc}}(V;\mathbb{R}^m)$ for all $s \in (\underline{p}_0, \overline{p}_0)$. Moreover, for a vector $b \in \mathbb{R}^m$ and a test function $\varphi \in C_0^\infty(V)$ we have the homogeneous Caccioppoli-type inequality*

$$\|\varphi(\cdot)v'(\cdot)\|_{L^s(V;\mathbb{R}^{m\times n})} \leqslant C_0\|(v(\cdot)-b)\otimes\varphi'(\cdot)\|_{L^s(V;\mathbb{R}^{m\times n})} \tag{9}$$

with some constant $C_0 = C_0(F, G, K, s) > 0$ depended only F, G, K, and s.

Now we apply Theorem 2.1 to improve some results from [19].

Let $n, m, k \in \mathbb{N}$ such that $2 \leqslant k \leqslant \min\{n, m\}$. We need the following hypothesis on continuous functions $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ and $G\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ (see [19]):

(H1) F is quasiconvex;

(H1′) F is strictly k-quasiconvex;

(H2) G is a null Lagrangian;

(H3) F and G are positively homogeneous of degree k;

(H4) $\sup\{K \geqslant 0 : F(\zeta) \geqslant KG(\zeta),\ \zeta \in \mathbb{R}^{m\times n}\} = 1$;

(H5) $c_F^0 := \inf\{F(\zeta) : \zeta \in \mathbb{R}^{m\times n},\ |\zeta| = 1\} > 0$;

(H6) $d_G := \sup\bigl\{\sum_{J\in\Gamma_m^k, I\in\Gamma_n^k} |\gamma_{JI}||x_I|^2 : x \in \mathbb{R}^n,\ |x| = 1\bigr\} < kc_F^0/(n-k)$ in the case $k < n$.

Here the coefficients γ_{JI} are taken from (6) for the null Lagrangian G. By (H3), the representation (6) for the null Lagrangian G consists only of $k \times k$-minors; i. e.,

$$G(\zeta) = \sum_{J\in\Gamma_m^k, I\in\Gamma_n^k} \gamma_{JI} \det\nolimits_{JI} \zeta, \quad \zeta \in \mathbb{R}^{m\times n}. \tag{10}$$

Since F is continuous, (H3) implies (7) with $c_F = c_F^0$.

The following theorem on Hölder regularity of solutions to (1) is a straightforward consequence of Theorem 2.1 and [19, Theorem 8].

Theorem 2.3 (Hölder regularity). *Suppose F and G satisfy* (H2)–(H6). *Put $K_0 = \infty$ for $k = n$ and $K_0 = \frac{kc_F^0}{(n-k)d_G}$ for $k < n$. Suppose that $K \in [1, K_0)$ and $\delta \in (0, 1)$ satisfy the inequality*

$$\frac{Kd_G}{kc_F^0} \leqslant \frac{1}{n-k+k\delta}. \tag{11}$$

Let $\underline{p}$ is the exponent from Theorem 2.1, *$p > \underline{p}$, and V be an open set in $\mathbb{R}^n$. Then every solution $v \in W^{1,p}_{\mathrm{loc}}(V; \mathbb{R}^m)$ to* (1) *with $H(x) \equiv \mathrm{const}$ satisfies the Hölder condition with exponent δ on each compact subset in V.*

The next theorem on stability of the class of solutions to (3) is a straightforward consequence of Corollary 2.2 and [19, Theorem 4].

Theorem 2.4 (Stability in the C-norme). *Suppose that F and G satisfy* (H1)–(H6). *Let $K \geqslant 1$, and let $\underline{p}_0$ denote the exponent from Corollary* 2.2. *Let V be a domain in $\mathbb{R}^n$, and let U be a compact subset in V. Then there is a function $\alpha(K) = \alpha_{F,G,V,U}(K)$ defined for $1 \leqslant K < K_0$ and such that $\lim_{K\to 1} \alpha(K) = \alpha(1) = 0$ and, for each mapping $v \in W^{1,p}_{\mathrm{loc}}(V; \mathbb{R}^m)$, $p > \underline{p}_0$, which satisfies inequality* (2) *there is a mapping $u \in W^{1,k}_{\mathrm{loc}}(V; \mathbb{R}^m)$ which is a solution to* (3) *such that*

$$\|v - u\|_{C(U;\mathbb{R}^m)} \leqslant \alpha(K) \operatorname{diam} v(V). \tag{12}$$

The following theorem improves Theorem 2.4 in the case when the function F satisfies (H1′). In this case, in addition to the estimate (12) of proximity (in the C-norm) of solutions of inequality (1) to solutions to equation (3), we obtain proximity estimates (in the L^k-norm) for the derivatives of these mappings. The theorem is a straightforward consequence of Corollary 2.2 and [19, Theorem 6].

Theorem 2.4 (Stability in the Sobolev norme). *Suppose that F and G satisfy* (H1′) *and* (H2)–(H6). *Then the conclusion of Theorem* 2.4 *is valid together with* (12) *and the following inequality*:

$$\|v' - u'\|_{L^k(U;\mathbb{R}^{m\times n})} \leqslant \alpha(K) \operatorname{diam} v(V). \tag{13}$$

3. Auxiliary Lemmas

In the proof of the higher integrability theorem we need the following lemmas.

Lemma 3.1. *Let $n, m, k \in \mathbb{N}$, $2 \leqslant k \leqslant \min(m,n)$, $p \geqslant 1$, $b \in \mathbb{R}^m$, $I = (i_1, \dots, i_k) \in \Gamma_n^k$, $J = (j_1, \dots, j_k) \in \Gamma_m^k$, and $V \subset \mathbb{R}^n$ be an open set. For a mapping $v\colon V \to \mathbb{R}^m$ and a function $\varphi\colon V \to \mathbb{R}$ define the mappings $w = (w_1, \dots, w_m)\colon V \to \mathbb{R}^m$ and $h = (h_1, \dots, h_m)\colon V \to \mathbb{R}^m$ by the rules $w(\cdot) := v(\cdot) - b$ and $h(\cdot) := \varphi(\cdot)w(\cdot)$. If v and φ are differentiable at $x \in V$, then the following inequality*

$$\Big||\varphi(x)v'(x)|^{p-k}\det{}_{JI}(\varphi(x)v'(x)) - |h'(x)|^{p-k}\det{}_{JI}h'(x)\Big| \\ \leqslant (2k+p)|w(x)\otimes\varphi'(x)|(|h'(x)| + |w(x)\otimes\varphi'(x)|)^{p-1} \quad (14)$$

holds.

$\lhd$ Observe that

$$\varphi(x)v'(x) = h'(x) - w(x)\otimes\varphi'(x). \quad (15)$$

We have

$$\det{}_{JI}(h' - w\otimes\varphi') = \det\begin{pmatrix} \partial_I h_{j_1} - w_{j_1}\partial_I\varphi \\ \vdots \\ \partial_I h_{j_k} - w_{j_k}\partial_I\varphi \end{pmatrix}.$$

Hence, $\det_{JI}(h' - w\otimes\varphi') = \det_{JI}h' - \sum_{\varkappa=1}^{k} A_\varkappa$ where

$$A_\varkappa := \det\begin{pmatrix} \partial_I h_{j_1} \\ \vdots \\ \partial_I h_{j_{\varkappa-1}} \\ w_{j_\varkappa}\partial_I\varphi \\ \partial_I h_{j_{\varkappa+1}} - w_{j_{\varkappa+1}}\partial_I\varphi \\ \vdots \\ \partial_I h_{j_k} - w_{j_k}\partial_I\varphi \end{pmatrix}.$$

Observe that

$$|A_\varkappa| \leqslant |\partial_I h_{j_1}| \dots |\partial_I h_{j_{\varkappa-1}}|\, |w_{j_\varkappa}\partial_I\varphi|\, |\partial_I h_{j_{\varkappa+1}} - w_{j_{\varkappa+1}}\partial_I\varphi| \dots |\partial_I h_{j_k} - w_{j_k}\partial_I\varphi| \\ \leqslant |w_J\otimes\partial_I\varphi|\, |\partial_I h_J|^{\varkappa-1}|\partial_I h_J - w_J\otimes\partial_I\varphi|^{k-\varkappa} \leqslant |w\otimes\varphi'|\,|h'|^{\varkappa-1}|h' - w\otimes\varphi'|^{k-\varkappa}.$$

Therefore,

$$|\det{}_{JI}(h' - \varphi v') - \det{}_{JI}h'| \leqslant |w\otimes\varphi'|\sum_{\varkappa=1}^{k}|h'|^{\varkappa-1}|h' - w\otimes\varphi'|^{k-\varkappa}. \quad (16)$$

Let us consider 2 cases.

Case 1:

$$0 < |h'(x) - w(x)\otimes\varphi'(x)| \leqslant |h'(x)|. \quad (17)$$

We have

$$\big||h' - w\otimes\varphi'|^{p-k}\det{}_{JI}(h' - w\otimes\varphi') - |h'|^{p-k}\det{}_{JI}h'\big| \\ \leqslant |h' - w\otimes\varphi'|^{p}\big|\det{}_{JI}(h' - w\otimes\varphi')\big|\,\big||h' - w\otimes\varphi'|^{-k} - |h'|^{-k}\big| \\ + |h' - w\otimes\varphi'|^{p}|h'|^{-k}\big|\det{}_{JI}(h' - w\otimes\varphi') - \det{}_{JI}h'\big| \\ + |h'|^{-k}|\det{}_{JI}h'|\,\big||h' - w\otimes\varphi'|^{p} - |h'|^{p}\big|. \quad (18)$$

We estimate each term on the right-hand side of (18). We obviously have the inequalities

$$a_1^{-k} - a_2^{-k} \leqslant k(a_2 - a_1)\, a_1^{-k-1} \tag{19}$$

and

$$a_2^{p} - a_1^{p} \leqslant p(a_2 - a_1)\, a_2^{p-1} \tag{20}$$

for $0 < a_1 \leqslant a_2$. Using (17) and (19), we get

$$\begin{aligned} \big||h' - w\otimes\varphi'|^{-k} - |h'|^{-k}\big| &= |h' - w\otimes\varphi'|^{-k} - |h'|^{-k} \\ &\leqslant k(|h'| - |h' - w\otimes\varphi'|)|h' - w\otimes\varphi'|^{-k-1} \leqslant k|w\otimes\varphi'|\,|h' - w\otimes\varphi'|^{-k-1}. \end{aligned} \tag{21}$$

By Hadamard's inequality

$$|\det{}_{JI}(h' - w\otimes\varphi')| \leqslant |h' - w\otimes\varphi'|^{k}. \tag{22}$$

Combining (21) with (22), we have

$$\begin{aligned} &|h' - w\otimes\varphi'|^{p}|\det{}_{JI}(h' - w\otimes\varphi')|\,\big||h' - w\otimes\varphi'|^{-k} - |h'|^{-k}\big| \\ &\qquad\leqslant k|w\otimes\varphi'|\,|h' - w\otimes\varphi'|^{p-1} \leqslant k|w\otimes\varphi'|(|h'| + |w\otimes\varphi'|)^{p-1}. \end{aligned} \tag{23}$$

Using (16) and (17), we obtain

$$\begin{aligned} &|h' - w\otimes\varphi'|^{p}|h'|^{-k}|\det{}_{JI}(h' - w\otimes\varphi') - \det{}_{JI} h'| \\ &\qquad\leqslant |h' - w\otimes\varphi'|^{p}|h'|^{-k}|w\otimes\varphi'|\sum_{\varkappa=1}^{k}|h'|^{\varkappa-1}|h' - w\otimes\varphi'|^{k-\varkappa} \\ &\qquad\leqslant |w\otimes\varphi'|\,|h' - w\otimes\varphi'|^{p-1}\sum_{\varkappa=1}^{k}(|h' - w\otimes\varphi'|/|h'|)^{k-\varkappa+1} \\ &\qquad\leqslant k|w\otimes\varphi'|\,|h' - w\otimes\varphi'|^{p-1} \leqslant k|w\otimes\varphi'|(|h'| + |w\otimes\varphi'|)^{p-1}. \end{aligned} \tag{24}$$

Using (17) and (20), we get

$$\begin{aligned} \big||h' - w\otimes\varphi'|^{p} - |h'|^{p}\big| &= |h'|^{p} - |h' - w\otimes\varphi'|^{p} \\ &\leqslant p(|h'| - |h' - w\otimes\varphi'|)|h'|^{p-1} \leqslant p|w\otimes\varphi'|\,|h'|^{p-1}. \end{aligned} \tag{25}$$

By Hadamard's inequality

$$|\det{}_{JI} h'| \leqslant |h'|^{k}. \tag{26}$$

Combining (25) with (26), we have

$$\begin{aligned} |h'|^{-k}|\det{}_{JI} h'|\,||h' - w\otimes\varphi'|^{p} - |h'|^{p}| &\leqslant p|w\otimes\varphi'|\,|h'|^{p-1} \\ &\leqslant p|w\otimes\varphi'|(|h'| + |w\otimes\varphi'|)^{p-1}. \end{aligned} \tag{27}$$

Using (15), (18), (23), (24), and (27), we obtain (14) in case 1.

Case 2:

$$0 < |h'(x)| \leqslant |h'(x) - w(x)\otimes\varphi'(x)|. \tag{28}$$

We have

$$\begin{aligned}
&||h' - w \otimes \varphi'|^{p-k} \det_{JI}(h' - w \otimes \varphi') - |h'|^{p-k} \det_{JI} h'| \\
&\qquad \leqslant |h' - w \otimes \varphi'|^{-k} |\det_{JI}(h' - w \otimes \varphi')| \, \big||h' - w \otimes \varphi'|^p - |h'|^p\big| \\
&\qquad\quad + |h' - w \otimes \varphi'|^{-k} |h'|^p \big|\det_{JI}(h' - w \otimes \varphi') - \det_{JI} h'\big| \\
&\qquad\quad + |h'|^p |\det_{JI} h'| \, \big||h' - w \otimes \varphi'|^{-k} - |h'|^{-k}\big|. \qquad (29)
\end{aligned}$$

We estimate each term on the right-hand side of (29). Using (20) and (28), we get

$$\begin{aligned}
&||h' - w \otimes \varphi'|^p - |h'|^p| = |h' - w \otimes \varphi'|^p - |h'|^p \\
&\qquad \leqslant p(|h' - w \otimes \varphi'| - |h'|)|h' - w \otimes \varphi'|^{p-1} \leqslant p|w \otimes \varphi'| \, |h' - w \otimes \varphi'|^{p-1}.
\end{aligned}$$

If we combain this with (22), we obtain

$$\begin{aligned}
&|h' - w \otimes \varphi'|^{-k} |\det_{JI}(h' - w \otimes \varphi')| \, \big||h' - w \otimes \varphi'|^p - |h'|^p\big| \\
&\qquad \leqslant p|w \otimes \varphi'| \, |h' - w \otimes \varphi'|^{p-1} \leqslant p|w \otimes \varphi'|(|h'| + |w \otimes \varphi'|)^{p-1}. \qquad (30)
\end{aligned}$$

Using (16) and (28), we get

$$\begin{aligned}
&|h' - w \otimes \varphi'|^{-k} |h'|^p |\det_{JI}(h' - w \otimes \varphi') - \det_{JI} h'| \\
&\qquad \leqslant |h' - w \otimes \varphi'|^{-k} |h'|^p |w \otimes \varphi'| \sum_{\varkappa=1}^{k} |h'|^{\varkappa-1} |h' - w \otimes \varphi'|^{k-\varkappa} \\
&\qquad \leqslant |w \otimes \varphi'| \, |h'|^{p-1} \sum_{\varkappa=1}^{k} (|h'|/|h' - w \otimes \varphi'|)^{\varkappa} \\
&\qquad \leqslant k|w \otimes \varphi'| \, |h'|^{p-1} \leqslant k|w \otimes \varphi'|(|h'| + |w \otimes \varphi'|)^{p-1}. \qquad (31)
\end{aligned}$$

Using (19) and (28), we have

$$\begin{aligned}
&\big||h' - w \otimes \varphi'|^{-k} - |h'|^{-k}\big| = |h'|^{-k} - |h' - w \otimes \varphi'|^{-k} \\
&\qquad \leqslant k(|h' - w \otimes \varphi'| - |h'|)|h'|^{-k-1} \leqslant k|w \otimes \varphi'| \, |h'|^{-k-1}.
\end{aligned}$$

If we combain this with (26), we obtain

$$\begin{aligned}
&|h'|^p |\det_{JI} h'| \, ||h' - w \otimes \varphi'|^{-k} - |h'|^{-k}| \leqslant k|w \otimes \varphi'| \, |h'|^{p-1} \\
&\qquad \leqslant k|w \otimes \varphi'|(|h'| + |w \otimes \varphi'|)^{p-1}. \qquad (32)
\end{aligned}$$

Combining (15) and (29)–(32), we get (14) in case 2. ▷

Lemma 3.2. *Suppose that a null Lagrangian G is homogeneous of degree k. Under the conditions of Lemma* 3.1, *we have*

$$\begin{aligned}
&||\varphi(x)v'(x)|^{p-k} G(\varphi(x)v'(x)) - |h'(x)|^{p-k} G(h'(x))| \\
&\qquad \leqslant C|w(x) \otimes \varphi'(x)|(|h'(x)| + |w(x) \otimes \varphi'(x)|)^{p-1} \qquad (33)
\end{aligned}$$

with some constant $C = C(G, p)$ depended only on G and p.

◁ Using (10) and (14), we get

$$
\begin{aligned}
&||\varphi(x)v'(x)|^{p-k}G(\varphi(x)v'(x)) - |h'(x)|^{p-k}G(h'(x))| \\
&\qquad\leqslant \sum_{J\in\Gamma^k_m, I\in\Gamma^k_n} |\gamma_{JI}|\,||\varphi(x)v'(x)|^{p-k}\det\nolimits_{JI}(\varphi(x)v'(x)) - |h'(x)|^{p-k}\det\nolimits_{JI} h'(x)| \\
&\qquad\qquad\leqslant (2k+p)\Bigg(\sum_{J\in\Gamma^k_m, I\in\Gamma^k_n} |\gamma_{JI}|\Bigg)|w(x)\otimes\varphi'(x)|(|h'(x)| + |w(x)\otimes\varphi'(x)|)^{p-1}. \triangleright
\end{aligned}
$$

Remark 3.3. We can use $C(G,p) = (2k+p)\sum_{J\in\Gamma^k_m, I\in\Gamma^k_n}|\gamma_{JI}|$ as the constant in (33). Here γ_{JI} are the coefficients from (10).

Also, we need the following version of Gehring's lemma (for example, see [36, Corollary 14.3.1]):

Lemma 3.4. *Suppose f and g are non-negative functions of the class $L^q(\mathbb{R}^n)$, $1 < q < \infty$, and satisfy*

$$
\Bigg(\frac{1}{|B(a,R)|}\int\limits_{B(a,R)} f^q\Bigg)^{1/q} \leqslant \frac{A}{|B(a,2R)|}\int\limits_{B(a,2R)} f + \Bigg(\frac{1}{|B(a,2R)|}\int\limits_{B(a,2R)} g^q\Bigg)^{1/q}
$$

for all balls $B(a,R) \subset \mathbb{R}^n$ and some constant $A > 0$. Then the inequality

$$
\int f^{q'} \leqslant C\int g^{q'}.
$$

holds with some exponent $q' = q'(n,q,A) > q$ and some constant $C = C(n,q,A) > 0$ depended only n, q, and A.

4. Proof of the Higher Intagrability Theorem

We are now in a position to prove the higher intagrability theorem.

◁ Proof of Theorem 2.1. Let $V \subset \mathbb{R}^n$ be an open set and $1 < p \leqslant t$. A suitable range of Sobolev exponents p will be defined below (see (46), (47), and (48)).

Fix a solution $v \in W^{1,p}_{\mathrm{loc}}(V;\mathbb{R}^m)$ to (1). Consider $\varphi \in C^\infty_0(V)$. We may assume that φ is non-negative as otherwise we could consider $|\varphi|$ which has not effect on the required Caccioppoli-type inequality (8). Using (1) and $F(\zeta) \geqslant c_F|\zeta|^k$, we get

$$
|v'(x)|^k \leqslant c_F^{-1}KG(v'(x)) + c_F^{-1}H_+(x) \quad \text{a.e. } V. \tag{34}
$$

Multipling both sides by $\varphi^p(x)|v'(x)|^{p-k}$ and using k-homogeneity of G, we obtain

$$
\begin{aligned}
|\varphi(x)v'(x)|^p \leqslant c_F^{-1}K|\varphi(x)v'(x)|^{p-k}G(\varphi(x)v'(x)) \\
+ c_F^{-1}\varphi^p(x)|v'(x)|^{p-k}H_+(x) \quad \text{a.e. } V.
\end{aligned} \tag{35}
$$

Let $\varepsilon > 0$. Put $V_1 := \{x \in V : |v'(x)| \geqslant \varepsilon\}$ and $V_2 := \{x \in V : |v'(x)| < \varepsilon\}$. We have $V_1 \cap V_2 = \varnothing$ and $|V \setminus (V_1 \cup V_2)| = 0$. Then

$$
\int\limits_V |\varphi v'|^p = \int\limits_{V_1} |\varphi v'|^p + \int\limits_{V_2} |\varphi v'|^p. \tag{36}
$$

We estimate each term on the right-hand side of (36).

Using (35), we have

$$\int\limits_{V_1} |\varphi v'|^p \leqslant c_F^{-1} K \int\limits_{V_1} |\varphi v'|^{p-k} G(\varphi v') + c_F^{-1} \int\limits_{V_1} \varphi^p |v'|^{p-k} H_+. \tag{37}$$

We estimate each term on the right-hand side of (37).

Using (10) and Hadamard's inequality, we have $-G(\zeta) \leqslant |G(\zeta)| \leqslant C_1(G)|\zeta|^k$, $\zeta \in \mathbb{R}^{m\times n}$, with $C_1(G) := \sum_{J\in\Gamma^k_m, I\in\Gamma^k_n} |\gamma_{JI}|$. Here γ_{JI} are the coefficients from (10). Then $|\zeta|^{p-k}G(\zeta) + C_1(G)|\zeta|^p \geqslant 0$. Therefore,

$$\int\limits_{V_2} |\varphi v'|^{p-k} G(\varphi v') + C_1(G) \int\limits_{V_2} |\varphi v'|^p \geqslant 0.$$

Consequently,

$$\begin{aligned}\int\limits_{V_1} |\varphi v'|^{p-k} G(\varphi v') \leqslant \int\limits_{V_1} |\varphi v'|^{p-k} G(\varphi v') + \int\limits_{V_2} |\varphi v'|^{p-k} G(\varphi v') \\ + C_1(G) \int\limits_{V_2} |\varphi v'|^p = \int\limits_{V} |\varphi v'|^{p-k} G(\varphi v') + C_1(G) \int\limits_{V_2} |\varphi v'|^p. \end{aligned} \tag{38}$$

We estimate each term on the right-hand side of (38).

Let $b \in \mathbb{R}^m$. Define the auxiliary mappings $w\colon V \to \mathbb{R}^m$ and $h\colon \mathbb{R}^n \to \mathbb{R}^m$ by the rules $w(x) := v(x) - b$, $x \in V$, and

$$h(x) := \begin{cases} \varphi(x) w(x), & x \in V, \\ 0, & x \in \mathbb{R}^n \setminus V. \end{cases}$$

Then $h \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$. We have $\varphi v' = h' - w \otimes \varphi'$.

Successively using Lemma 3.2, (10), and [22, Theorem 2.1], we deduce

$$\begin{aligned}\int\limits_{V} |\varphi v'|^{p-k} G(\varphi v') &\leqslant \int |h'|^{p-k} G(h') + C(G,p) \int\limits_{V} (|h'| + |w \otimes \varphi'|)^{p-1} |w \otimes \varphi'| \\ &= \sum_{J\in\Gamma^k_m, I\in\Gamma^k_n} \gamma_{JI} \int |h'|^{p-k} \det\nolimits_{JI}(h') + C(G,p) \int\limits_{V} (|h'| + |w \otimes \varphi'|)^{p-1} |w \otimes \varphi'| \\ &\leqslant C_2(G)|1 - p/k| \int |h'|^p + C(G,p) \int\limits_{V} (|h'| + |w \otimes \varphi'|)^{p-1} |w \otimes \varphi'|, \end{aligned} \tag{39}$$

where $C_2(G) := C(k)C_1(G)$, $C(k)$ is from [22, Theorem 2.1], and $C(G,p) = (2k+p)C_1(G)$ is from Remark 3.3.

Since $|v'| < \varepsilon$ on V_2, we have

$$\int\limits_{V_2} |\varphi v'|^p \leqslant \varepsilon \int\limits_{V_2} |\varphi v'|^{p-1} \varphi. \tag{40}$$

Combining (38), (39), and (40), we get

$$\int\limits_{V_1} |\varphi v'|^{p-k} G(\varphi v') \leqslant C_2(G)|1-p/k| \int |h'|^p + C(G,p)\int\limits_{V} (|h'| + |w\otimes\varphi'|)^{p-1}|w\otimes\varphi'| + \varepsilon C_1(G)\int\limits_{V_2} |\varphi v'|^{p-1}\varphi. \quad (41)$$

Since $|v'| \geqslant \varepsilon$ on V_1, we have

$$\int\limits_{V_1} \varphi^p |v'|^{p-k} H_+ \leqslant \varepsilon^{1-k} \int\limits_{V_1} |\varphi v'|^{p-1}\varphi H_+. \quad (42)$$

Using (37), (41), and (42), we obtain

$$\int\limits_{V_1} |\varphi v'|^p \leqslant c_F^{-1} K C_2(G)|1-p/k| \int |h'|^p + c_F^{-1}KC(G,p)\int\limits_{V} (|h'| + |w\otimes\varphi'|)^{p-1}|w\otimes\varphi'| + \varepsilon c_F^{-1}KC_1(G)\int\limits_{V_2} |\varphi v'|^{p-1}\varphi + \varepsilon^{1-k}c_F^{-1}\int\limits_{V_1} |\varphi v'|^{p-1}\varphi H_+. \quad (43)$$

Now combining (36), (40), and (43), and using $|\varphi v'| \leqslant |h'| + |w\otimes\varphi'|$, we get

$$\begin{aligned}\int\limits_{V} |\varphi v'|^p &\leqslant c_F^{-1} K C_2(G)|1-p/k| \int |h'|^p \\ &+ c_F^{-1}KC(G,p)\int\limits_{V} (|h'| + |w\otimes\varphi'|)^{p-1}|w\otimes\varphi'| \\ &+ \varepsilon(c_F^{-1}KC_1(G)+1)\int\limits_{V_2} |\varphi v'|^{p-1}\varphi + \varepsilon^{1-k}c_F^{-1}\int\limits_{V_1} |\varphi v'|^{p-1}\varphi H_+ \quad (44)\\ &\leqslant c_F^{-1} K C_2(G)|1-p/k| \int |h'|^p + [c_F^{-1}K(C(G,p)+C_1(G)+1)+1] \\ &\times \int\limits_{V} (|h'| + |w\otimes\varphi'|)^{p-1}(|w\otimes\varphi'| + \varphi(\varepsilon + \varepsilon^{1-k}H_+)).\end{aligned}$$

Observe that $|h'|^p \leqslant 2^{p-1}(|\varphi v'|^p + |w\otimes\varphi'|^p)$. By (44), we get

$$\int |h'|^p \leqslant 2^{p-1}\int\limits_{V} |\varphi v'|^p + 2^{p-1}\int\limits_{V} |w\otimes\varphi'|^p \leqslant 2^{p-1}c_F^{-1}KC_2(G)|1-p/k|\int |h'|^p + 2^{p-1}\big[c_F^{-1}K(C(G,p)+C_1(G)+1)+2\big] \times \int\limits_{V} (|h'| + |w\otimes\varphi'|)^{p-1}(|w\otimes\varphi'| + \varphi(\varepsilon + \varepsilon^{1-k}H_+)). \quad (45)$$

Put

$$\underline{p} = \underline{p}(F,G,K) := \inf\left\{p \geqslant 1 : 2^{p-1}c_F^{-1}KC_2(G)|1-p/k| < 1\right\} \tag{46}$$

and

$$\overline{p} = \overline{p}(F,G,K,t) := \sup\left\{p \leqslant t : 2^{p-1}c_F^{-1}KC_2(G)|1-p/k| < 1\right\}. \tag{47}$$

For

$$p \in (\underline{p}, \overline{p}) \tag{48}$$

we have $2^{p-1}c_F^{-1}KC_2(G)|1-p/k| < 1$. From (45) we derive

$$\int |h'|^p \leqslant C_3(F,G,K,p) \int\limits_V (|h'| + |w \otimes \varphi'|)^{p-1}(|w \otimes \varphi'| + \varphi(\varepsilon + \varepsilon^{1-k}H_+)), \tag{49}$$

where $C_3(F,G,K,p) := \frac{2^{p-1}[c_F^{-1}K(C(G,p)+C_1(G)+1)+2]}{1-2^{p-1}c_F^{-1}KC_2(G)|1-p/k|}$.

Successively using $(|h'|+|w\otimes\varphi'|)^p \leqslant 2^{p-1}(|h'|^p+|w\otimes\varphi'|^p)$, (49), and Hölder's inequality, we obtain

$$\begin{aligned}
\int\limits_V (|h'| + |w\otimes\varphi'|)^p &\leqslant 2^{p-1}\left(\int |h'|^p + \int\limits_V |w\otimes\varphi'|^p\right)\\
&\leqslant C(F,G,K,p)\int\limits_V (|h'| + |w\otimes\varphi'|)^{p-1}(|w\otimes\varphi'| + \varphi(\varepsilon+\varepsilon^{1-k}H_+))\\
&\leqslant C(F,G,K,p)\left(\int\limits_V (|h'| + |w\otimes\varphi'|)^p\right)^{\frac{1}{p-1}}\left(\int\limits_V (|w\otimes\varphi'| + \varphi(\varepsilon+\varepsilon^{1-k}H_+))^p\right)^{\frac{1}{p}},
\end{aligned}$$

where $C(F,G,K,p) := 2^{p-1}(C_3(F,G,K,p)+1)$. Therefore,

$$\||h'| + |w\otimes\varphi'|\|_{L^p(V)} \leqslant C(F,G,K,p)\||w\otimes\varphi'| + \varphi(\varepsilon+\varepsilon^{1-k}H_+)\|_{L^p(V)}.$$

Using $|\varphi v'| \leqslant |h'| + |w\otimes\varphi'|$ and $w = v - b$, we get the Caccioppoli-type estimate

$$\|\varphi v'\|_{L^p(V;\mathbb{R}^{m\times n})} \leqslant C(F,G,K,p)\||(v-b)\otimes\varphi'| + \varphi(\varepsilon+\varepsilon^{1-k}H_+)\|_{L^p(V)}. \tag{50}$$

Of course now we observe that this inequality holds with p replaced by s for any $s \in (\underline{p}, \overline{p})$, provided we know *a priori* that $v \in W^{1,s}_{\mathrm{loc}}(V;\mathbb{R}^m)$.

Let $S = \{s \in (\underline{p},\overline{p}) : v \in W^{1,s}_{\mathrm{loc}}(V;\mathbb{R}^m)\}$. We have $p \in S$. Therefore, $S \neq \varnothing$. For $s \in S$ we have (8); the constant $C = C(F,G,K,s)$ which depends continuously on s is finite in the range $(\underline{p},\overline{p})$ but may blow up at the endpoints. If we combine this with the Sobolev embedding theorem, we obtain that S is relatively closed in $(\underline{p},\overline{p})$. The theorem will be proved if we can show that S is open. Obviously, if $s \in S$, then $(\underline{p}, s] \subset (\underline{p},\overline{p})$. We are therefore left only with the task of showing higher integrability of the differential. It is at this point that Gehring's lemma comes to the rescue. Using (50), we easily derive reverse Hölder inequalities for v'.

Consider $x_0 \in V$ and $0 < R < \mathrm{dist}(x_0, V)/3$. Put $B_R := B(x_0,R)$ and $B_{2R} := B(x_0,2R)$. Let $\eta \in C_0^\infty(B_{2R})$ be a nonnegative function such that

$$\eta = 1 \text{ on } B_R, \quad 0 \leqslant \eta \leqslant 1 \text{ and } |\eta'| \leqslant \frac{C(n)}{R} \text{ on } B_{2R}. \tag{51}$$

Substituting φ by η and b by $v_{B_{2R}} := \frac{1}{|B_{2R}|}\int_{B_{2R}} v$ in (50), we get

$$\|\eta v'\|_{L^p(B_{2R};\mathbb{R}^{m\times n})} \leqslant C(F,G,K,p)\||(v-v_{B_{2R}})\otimes\eta'|+\eta(\varepsilon+\varepsilon^{1-k}H_+)\|_{L^p(B_{2R})}.$$

Therefore,

$$\int\limits_{B_{2R}} |\eta v'|^p \leqslant C_4(F,G,K,p)\biggl(\int\limits_{B_{2R}} |(v-v_{B_{2R}})\otimes\eta'|^p + \int\limits_{B_{2R}} \eta(\varepsilon+\varepsilon^{1-k}H_+)^p\biggr),$$

where $C_4(F,G,K,p) = (C(F,G,K,p))^p$. Successively using (51) and the Poincaré–Sobolev inequality (for example, see [36, Theorem 4.10.3]), we obtain

$$\int\limits_{B_R} |v'|^p \leqslant C_5(F,G,K,p)\Biggl(R^{-p}\int\limits_{B_{2R}} |v-v_{B_{2R}}|^p + \int\limits_{B_{2R}} \eta(\varepsilon+\varepsilon^{1-k}H_+)^p\Biggr)$$

$$\leqslant C_6(F,G,K,p)\Biggl(R^{-p}\biggl(\int\limits_{B_{2R}} |v'|^{\frac{np}{n+p}}\biggr)^{\frac{n+p}{n}} + \int\limits_{B_{2R}} (\varepsilon+\varepsilon^{1-k}H_+)^p\Biggr)$$

with some consants $C_5(F,G,K,p)$ and $C_6(F,G,K,p)$. Therefore,

$$\frac{1}{|B_R|}\int\limits_{B_R} |v'|^p \leqslant C_7(F,G,K,p)\Biggl(\biggl(\frac{1}{|B_{2R}|}\int\limits_{B_{2R}} |v'|^{\frac{np}{n+p}}\biggr)^{\frac{n+p}{n}} + \frac{1}{|B_{2R}|}\int\limits_{B_{2R}} (\varepsilon+\varepsilon^{1-k}H_+)^p\Biggr).$$

Hence, we have the reverse Hölder inequality

$$\biggl(\frac{1}{|B_R|}\int\limits_{B_R} |v'|^p\biggr)^{\frac{n}{n+p}} \leqslant \frac{C_8(F,G,K,p)}{|B_{2R}|}\int\limits_{B_{2R}} |v'|^{\frac{np}{n+p}}$$

$$+\Biggl(\frac{1}{|B_{2R}|}\int\limits_{B_{2R}} \bigl(C_9(F,G,K,p)(\varepsilon+\varepsilon^{1-k}H_+)\bigr)^p\Biggr)^{\frac{n}{n+p}}.$$

Put $q = \frac{n+p}{n} > 1$, $f = |v'|^{\frac{np}{n+p}}$, and $g = (C_9(F,G,K,p)(\varepsilon+\varepsilon^{1-k}H_+))^{\frac{np}{n+p}}$. By Lemma 3.4 we conclude that f is integrable with a power slightly larger than q. This in turn means that v' is integrable with a slightly higher power then p and so $v \in W^{1,p'}_{\mathrm{loc}}(V;\mathbb{R}^m)$ for some $p' > p$. ▷

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РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНОГО НЕРАВЕНСТВА С НУЛЬ-ЛАГРАНЖИАНОМ: ПОВЫШАЮЩАЯСЯ ИНТЕГРИРУЕМОСТЬ И УСТРАНИМОСТЬ ОСОБЕННОСТЕЙ. I

Егоров А. А.

Целью настоящей статьи является установление свойства самоулучшающейся интегрируемости производных решений дифференциального неравенства с нуль-лагранжианом. Более точно, мы доказываем, что решение класса Соболева с показателем суммирумости, немного меньшим естественно определенного структурными предположениями на нуль-лагранжиан показателя, фактически принадлежит пространству Соболева с показателем суммируемости, немного большим естественного показателя. Мы также применяем это свойство, чтобы улучшить теоремы о гёльдеровой регулярности и об устойчивости из статьи [19].

Ключевые слова: нуль-лагранжиан, повышающаяся интегрирумость, самоулучшающаяся регулярность, гёльдерова регулярность, устойчивость классов отображений.