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A CHARACTERIZATION OF ORDER BOUNDED DISJOINTNESS PRESERVING BILINEAR OPERATORS

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The paper is aimed to characterize order bounded disjointness preserving bilinear operators in terms of their null-spaces. To this end the Boolean valued analysis approach is employed.

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Key words: Boolean valued representation, vector lattice, disjointness preserving operator.

It was observed and employed in [1, 2, 3] that a linear operator T from a vector lattice X to a Dedekind complete vector lattice Y is, in a sense, determined up to an orthomorphism from the family of the kernels of the *strata* πT of T with π ranging over all band projections on Y. Similar reasoning was involved in [4] to characterize order bounded disjointness preserving bilinear operators. Unfortunately, Theorem 3.4 in [4] is erroneous and this note aims to give correct statement and proof of this result. Unexplained terms can be found on the theory of vector lattices and order bounded operators, in [5, 6], on Boolean valued analysis machinery, in [7, 8].

In what follows X, Y, and Z are Archimedean vector lattices, Z^{u} is a universal completion of Z, and $B : X \times Y \to Z$ is a bilinear operator. We denote the Boolean algebra of band projections in X by $\mathbb{P}(X)$. Recall that a linear operator $T : X \to Y$ is said to be *disjointness preserving* if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in X$. A bilinear operator $B : X \times Y \to Z$ is called *disjointness preserving* (a *lattice bimorphism*) if the linear operators $B(x, \cdot) : y \mapsto B(x, y)$ $(y \in Y)$ and $B(\cdot, y) : x \mapsto B(x, y)$ $(x \in X)$ are disjointness preserving for all $x \in X$ and $y \in Y$ (lattice homomorphisms for all $x \in X_+$ and $y \in Y_+$). Denote $X_\pi := \bigcap\{\ker(\pi B(\cdot, y)) :\ y \in Y\}$ and $Y_\pi := \bigcap\{\ker(\pi B(x, \cdot)) :\ x \in X\}$. Clearly, X_π and Y_π are vector subspaces of X and Y, respectively. Now we state the main result of the note.

Theorem. *Assume that X, Y, and Z are vector lattices with Z having the projection property. For an order bounded bilinear operator $B : X \times Y \to Z$ the following assertions are equivalent:*

(1) *B is disjointness preserving.*

(2) *There are a band projection $\varrho \in \mathbb{P}(Z)$ and lattice homomorphisms $S : X \to Z^{\mathrm{u}}$ and $T : Y \to Z^{\mathrm{u}}$ such that $B(x, y) = \varrho S(x)T(y) - \varrho^\perp S(x)T(y)$ for all $(x, y) \in X \times Y$.*

(3) *For every $\pi \in \mathbb{P}(Z)$ the subspaces X_π and Y_π are order ideals respectively in X and Y, and the kernel of every stratum πB of B with $\pi \in \mathbb{P}(Z)$ is representable as*

$$\ker(\pi B) = \bigcup\big\{X_\sigma \times Y_\tau :\ \sigma, \tau \in \mathbb{P}(Z);\ \sigma \vee \tau = \pi\big\}.$$

The proof presented below follows along general lines of [1–4]: Using the canonical embedding and ascent to the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we reduce the matter to characterizing disjointness preserving bilinear functional on the product of two vector lattices over dense subfield of the reals $\mathbb{R}$. The resulting scalar problem is solved by the following simple fact.

Lemma 1. *Let X and Y be vector lattices. For an order bounded bilinear functional $\beta : X \times Y \to \mathbb{R}$ the following assertions are equivalent:*

(i) *β is disjointness preserving.*

(ii) $\ker(\beta) = (X_0 \times Y) \cup (X \times Y_0)$ *for some order ideals $X_0 \subset X$ and $Y_0 \subset Y$.*

(iii) *There exist lattice homomorphisms $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ such that either $\beta(x, y) = g(x)h(y)$ or $\beta(x, y) = -g(x)h(y)$ for all $x \in X$ and $y \in Y$.*

$\lhd$ Assume that $\ker(\beta) = (X_0 \times Y) \cup (X \times Y_0)$ and take $y \in Y$. If $y \in Y_0$ then $\beta(\cdot, y) \equiv 0$, otherwise $\ker(\beta(\cdot, y)) = X_0$ and $\beta(\cdot, y)$ is disjointness preserving, since an order bounded linear functional is disjointness preserving if and only if its null-space is an order ideal. Similarly, $\beta(x, \cdot)$ is disjointness preserving for all $x \in X$ and thus $(ii) \implies (i)$. The implication $(i) \implies (iii)$ was established in [9, Theorem 3.2] and $(iii) \implies (i)$ is trivial with $X_0 = \ker(g)$ and $Y_0 = \ker(h)$. $\rhd$

Let $\mathbb{B}$ be a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ the corresponding Boolean valued model with Boolean truth values $[\![\varphi]\!]$ for set-theoretic formulas φ. There exists an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ which plays the role of a field of reals within $\mathbb{V}^{(\mathbb{B})}$. The descending functor sends every internal algebraic structure $\mathfrak{A}$ into its descent $\mathfrak{A}{\downarrow}$ which is an algebraic structure in conventional sense. Gordon's theorem (see [5, 8.1.2] and [10, Theoren 2.4.2]) tells us that the algebraic structure $\mathscr{R}{\downarrow}$ (with the descended operations and order relation) is an universally complete vector lattice. Moreover, there is a Boolean isomorphism χ of $\mathbb{B}$ onto $\mathbb{P}(\mathscr{R}{\downarrow})$ such that $b \leqslant [\![\, x = y \,]\!]$ if and only if $\chi(b)x = \chi(b)y$. We identify $\mathbb{B}$ with $\mathbb{P}(\mathscr{R}{\downarrow})$ and take χ to be $I_{\mathbb{B}}$.

Let $[X \times Y, \mathscr{R}{\downarrow}] \in \mathbb{V}$ and $[\![X^\wedge \times Y^\wedge, \mathscr{R}]\!] \in \mathbb{V}^{(\mathbb{B})}$ stand for the sets respectively of all maps from $X \times Y$ to $\mathscr{R}{\downarrow}$ and from $X^\wedge \times X^\wedge$ to $\mathscr{R}$ (within $\mathbb{V}^{(\mathbb{B})}$). The correspondences $f \mapsto f{\uparrow}$, the modified ascent, is a bijection between $[X \times Y, \mathscr{R}{\downarrow}]$ and $[\![X^\wedge \times Y^\wedge, \mathscr{R}\,]\!]$. Given $f \in [X, \mathscr{R}{\downarrow}]$, the internal map $f{\uparrow} \in [\![X^\wedge, \mathscr{R}]\!]$ is uniquely determined by the relation $[\![f{\uparrow}(x^\wedge) = f(x)]\!] = \mathbb{1}$ $(x \in X)$. Observe also that $\pi \leqslant [\![f{\uparrow}(x^\wedge) = \pi f(x)]\!]$ $(x \in X,\ \pi \in \mathbb{P}(\mathscr{R}{\downarrow}))$. This fact specifies for bilinear operators as follows.

Lemma 2. *Let $B : X \times Y \to Y$ be a bilinear operator and $\beta := B{\uparrow}$ its modified ascent. Then $\beta : X^\wedge \times Y^\wedge \to \mathscr{R}$ is a $\mathbb{R}^\wedge$-bilinear functional within $\mathbb{V}^{(\mathbb{B})}$. Moreover, B is order bounded and disjointness preserving if and only $[\![\,\beta$ is order bounded and disjointness preserving$\,]\!] = \mathbb{1}$.*

$\lhd$ The proof goes along similar lines to the proof of Theorem 3.3.3 in [10]. $\rhd$

Lemma 3. *Let B and β be as in Lemma* 2. *Then* $[\![\ker(B)^\wedge = \ker(\beta)]\!] = \mathbb{1}$.

$\lhd$ Using the above mentioned determining property of modified ascent and interpreting the formal definition $z \in \ker(\beta) \leftrightarrow (\exists\, x \in X^\wedge)(\exists\, y \in Y^\wedge)(z = (x, y) \wedge \beta(x, y) = 0)$, the proof is reduced to a straightforward calculation:

$$
\begin{aligned}
[\![z \in \ker(\beta)]\!] &= \bigvee_{x \in X,\, y \in Y} [\![z = (x^\wedge, y^\wedge) \wedge \beta(x^\wedge, y^\wedge) = 0]\!] \\
&= \bigvee_{(x,y) \in X \times Y} [\![z = (x, y)^\wedge \wedge (x, y)^\wedge \in \ker(B)^\wedge]\!]
\end{aligned}
$$

$$\leqslant [\![z \in \ker(B)^\wedge]\!] = \bigvee_{(x,y)\in X\times Y} [\![z = (x,y)^\wedge \wedge (x.y) \in \ker(B)]\!]$$
$$= \bigvee_{x\in X,\, y\in Y} [\![(z = (x^\wedge, y^\wedge) \wedge \beta(x^\wedge, y^\wedge) = 0]\!]$$
$$\leqslant [\![z \in \ker(\beta)]\!]. \ \rhd$$

Lemma 4. *Define $\mathscr{X}$ and $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$ by $\mathscr{X} := \bigcap\{\ker(\beta(\cdot, Y)) : y \in Y^\wedge\}$ and $\mathscr{Y} := \bigcap\{\ker(\beta(x, \cdot)) : x \in X^\wedge\}$. Given arbitrary $\pi \in \mathbb{P}(Z)$, $x \in X$, and $y \in Y$, the equivalences hold:*

$$\pi \leqslant [\![x^\wedge \in \mathscr{X}]\!] \Longleftrightarrow x \in X_\pi, \quad \pi \leqslant [\![y^\wedge \in \mathscr{Y}]\!] \Longleftrightarrow y \in Y_\pi.$$

$\lhd$ For $\pi \in \mathbb{P}(Z)$ and $x \in X$ we need only to calculate Boolean truth values taking into account that $[\![B(x,y) = \beta(x^\wedge, v^\wedge)]\!] = \mathbb{1}$ for all $x \in X$ and $y \in Y$:

$$[\![x^\wedge \in \mathscr{X}]\!] = [\![(\forall\, v \in Y^\wedge)\beta(x^\wedge, v) = 0]\!] = \bigwedge_{v\in Y} [\![\beta(x^\wedge, v^\wedge) = 0]\!] = \bigwedge_{v\in Y} [\![B(x,v) = 0]\!].$$

It follows that $\pi \leqslant [\![x^\wedge \in \mathscr{X}]\!]$ if and only if $\pi \leqslant [\![B(x,v) = 0]\!]$ for all $v \in Y$. By Gorgon's theorem the latter means that $\pi B(x,v) = 0$ for all $v \in Y$, that is $x \in X_\pi$. $\rhd$

Lemma 5. *Let B and β be as in Lemma 2. For arbitrary $\pi \in \mathbb{P}(Z)$, $x \in X$, and $y \in Y$, we have $\pi \leqslant [\![(x^\wedge, y^\wedge) \in (\mathscr{X} \times Y) \cup (X \times \mathscr{Y})]\!]$ if and only if there exist $\sigma, \tau \in \mathbb{P}(Z)$ such that $\sigma \vee \tau = \pi$, $x \in X_\sigma$, and $y \in Y_\tau$.*

$\lhd$ Denote $\rho := [\![(x^\wedge, y^\wedge) \in (\mathscr{X} \times Y) \cup (X \times \mathscr{Y})]\!]$ and observe that

$$\rho = [\![(x^\wedge \in \mathscr{X}) \vee y^\wedge \in \mathscr{Y}]\!] = [\![x^\wedge \in \mathscr{X}]\!] \vee [\![y^\wedge \in \mathscr{Y}]\!].$$

Clearly, $\pi \leqslant \rho$ if and only if $\sigma \vee \tau = \pi$ for some $\sigma \leqslant [\![x^\wedge \in \mathscr{X}]\!]$ and $\tau \leqslant [\![y^\wedge \in \mathscr{Y}]\!]$, so that the required property follows from Lemma 4. $\rhd$

Proof of the main result. The implication (1) $\Longrightarrow$ (2) was proved in [9, Corollary 3.3], while (2) $\Longrightarrow$ (3) is straightforward. Indeed, observe first that if (2) is fulfilled then $|B(x,y)| = |B|(|x|,|y|) = |S|(|x|)|T|(|y|)$, so that we can assume S and T to be lattice homomorphisms, as in this event $\ker(B) = \ker(|B|)$. Take $\pi \in \mathbb{P}(Z)$ and denote $\sigma := \pi - \pi[Sx]$ and $\tau := \pi - \pi[Ty]$, where $[y]$ is a band projection onto $\{y\}^{\perp\perp}$. Observe next that $\pi B(x,y) = 0$ if and only if $\pi[Sx]$ and $\pi[Ty]$ are disjoint or, what is the same, if $\sigma \vee \tau = \pi$. Moreover, the map $\rho_y : x \mapsto \sigma S(x)T(y)$ is disjointness preserving for all $y \in Y$ and hence $X_\sigma = \bigcap_{y\in Y} \ker(\rho_y)$ is an order ideal in X. Similarly, Y_τ is an order ideal in Y. Thus, $(x,y) \in \ker(\pi B)$ if and only if $x \in X_\sigma$ and $y \in Y_\tau$ for some $\sigma, \tau \in \mathbb{P}(Z)$ with $\sigma \vee \tau = \pi$.

Prove the remaining implication (3) $\Longrightarrow$ (1). Suppose that for every $\pi \in \mathbb{P}(Y)$ the representation in (3) holds. Take $x, u \in X$ and put $\pi := [\![x^\wedge \in \mathscr{X}]\!]$, $\rho := [\![|u|^\wedge \leqslant |x|^\wedge]\!]$. By Lemma 4 we have $x \in X_\pi$. Note also that either $\rho = \mathbb{0}$ or $\rho = \mathbb{1}$. If $\rho = \mathbb{1}$ then $|u| \leqslant |x|$ and by hypotheses $u \in X_\pi$. Again by Lemma 4 we get $\rho \leqslant [\![u^\wedge \in \mathscr{X}]\!]$. This estimate is obvious whenever $\rho = \mathbb{0}$, so that $[\![x^\wedge \in \mathscr{X}]\!] \wedge [\![|u|^\wedge \leqslant |x|^\wedge]\!] \Rightarrow [\![u^\wedge \in \mathscr{X}]\!] = \mathbb{1}$ for all $x, u \in X$. Now, a simple calculation shows that $\mathscr{X}$ is an order ideal in $X^\wedge$:

$$[\![(\forall\, x, u \in X^\wedge)(|u| \leqslant |x| \,\wedge\, x \in \mathscr{X} \to u \in \mathscr{X})]\!]$$
$$= \bigwedge_{u,x\in X} \big([\![x \in \mathscr{X}]\!] \wedge [\![|u| \leqslant |x|]\!] \Rightarrow [\![u \in \mathscr{X}]\!]\big) = \mathbb{1}.$$

Similarly, $\mathscr{Y}$ is an order ideal in $Y^\wedge$.

It follows from the hypothesis (3) and Lemma 5 that $(x, y) \in \ker(\pi B)$ if and only if $\pi \leqslant [\![(x^\wedge, y^\wedge) \in (\mathscr{X} \times Y) \cup (X \times \mathscr{Y})]\!]$. Taking into account Lemma 2 and the observation made before it we conclude that $\pi \leqslant [\![(x^\wedge, y^\wedge) \in \ker(\beta)]\!])$ if and only if $\pi \leqslant [\![(x^\wedge, y^\wedge) \in (\mathscr{X} \times Y) \cup (X \times \mathscr{Y})]\!]$ and hence $[\![\ker(\beta) = (\mathscr{X} \times Y) \cup (X \times \mathscr{Y})]\!] = \mathbb{1}$. It remains to apply within $\mathbb{V}^{(\mathbb{B})}$ the equivalence (i)$\Longleftrightarrow$(iii) in Lemma 1. It follows that B is disjointness preserving according to Lemma 2. $\rhd$

Corollary. *Assume that Y has the projection property. An order bounded linear operator $T : X \to Y$ is disjointness preserving if and only if $\ker(bT)$ is an order ideal in X for every projection $b \in \mathbb{P}(Y)$.*

$\lhd$ Apply the above theorem to the bilinear operator $B : X \times \mathbb{R} \to Y$ defined as $B(x, \lambda) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$. $\rhd$

References

1. *Kutateladze S. S.* On differences of lattice homomorphisms // Sib. Math. J.—2005.—Vol. 46, № 2.—P. 393–396.
2. *Kutateladze S. S.* On Grothendieck subspaces // Sib. Math. J.—2005.—Vol. 46, No 3.—P. 620–624.
3. *Kutateladze S. S.* The Farkas lemma revisited // Sib. Math. J.—2010.—Vol. 51, No 1.—P. 78–87.
4. *Kusraev A. G., Kutateladze S. S.* On order bounded disjointness preserving operators // Sib. Math. J.—2014.—Vol. 55, № 5.—P. 915–928.
5. *Kusraev A. G.* Dominated Operators.—Dordrecht: Kluwer, 2000.
6. *Aliprantis C. D., Burkinshaw O.* Positive Operators.—N. Y.: Academic Press, 1985.—xvi+367 p.
7. *Bell J. L.* Boolean Valued Models and Independence Proofs in Set Theory.—N. Y.: Clarendon Press, 1985.—xx+165 p.
8. *Kusraev A. G., Kutateladze S. S.* Boolean Valued Analysis.—Novosibirsk: Nauka, 1999; Dordrecht: Kluwer, 1999.
9. *Kusraev A. G., Tabuev S. N.* On multiplicative representation of disjointness preserving bilinear operators // Sib. Math. J.—2008.—Vol. 49, № 2.—P. 357–366.
10. *Kusraev A. G., Kutateladze S. S.* Boolean Valued Analysis: Selected Topics.—Vladikavkaz: SMI VSC RAS, 2014.—iv+400 p.—(Trends in Science: The South of Russia. A Mathematical Monograph. Issue 6).

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О ХАРАТЕРИЗАЦИИ ПОРЯДКОВО ОГРАНИЧЕННЫХ БИЛИНЕЙНЫХ ОПЕРАТОРОВ, СОХРАНЯЮЩИХ ДИЗЪЮНКТНОСТЬ

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Цель заметки — дать характеризацию сохраняющих дизъюнктность порядково ограниченных билинейных операторов в векторных решетках в терминах ядер. В доказательстве основного результата используется булевозначный подход.

Ключевые слова: булевозначное представление, векторная решетка, сохраняющий дизъюнктность оператор.