1. Introduction

The Brezis-Lieb lemma [2, Theorem 2] has numerous applications mainly in calculus of variations (see, for example [3, 6]). We begin with its statement. Let \( j : \mathbb{C} \to \mathbb{C} \) be a continuous function with \( j(0) = 0 \). In addition, let \( j \) satisfy the following hypothesis: for every sufficiently small \( \varepsilon > 0 \), there exist two continuous, nonnegative functions \( \varphi_\varepsilon \) and \( \psi_\varepsilon \) such that

\[
|j(a + b) - j(a)| \leq \varepsilon \varphi_\varepsilon(a) + \psi_\varepsilon(b) \tag{1}
\]

for all \( a, b \in \mathbb{C} \). The following result has been stated and proved by H. Brezis and E. Lieb in [2].

**Theorem 1.1** (Brezis-Lieb lemma [2, Theorem 2]). Let \( (\Omega, \Sigma, \mu) \) be a measure space. Let the mapping \( j \) satisfy the above hypothesis, and let \( f_n = f + g_n \) be a sequence of measurable functions from \( \Omega \) to \( \mathbb{C} \) such that:

(i) \( g_n \xrightarrow{a.e.} 0 \);
(ii) \( j \circ f \in L^1 \);
(iii) \( \int \varphi_\varepsilon \circ g_n d\mu \leq C < \infty \) for some \( C \) independent of \( \varepsilon \) and \( n \);
(iv) \( \int \psi_\varepsilon \circ f d\mu < \infty \) for all \( \varepsilon > 0 \).

Then, as \( n \to \infty \),

\[
\int (j(f + g_n) - j(g_n) - j(f)) d\mu \to 0. \tag{2}
\]

Here we reproduce its proof from [2, Theorem 2] with several simple remarks.

< Fix \( \varepsilon > 0 \) and let \( W_{\varepsilon,n} = [\|j \circ f_n - j \circ g_n - j \circ f\| \leq \varepsilon \varphi_\varepsilon \circ g_n \]_. As \( n \to \infty \), \( W_{\varepsilon,n} \xrightarrow{a.e.} 0 \). On the other hand,

\[
|j \circ f_n - j \circ g_n - j \circ f| \leq |j \circ f_n - j \circ g_n| + |j \circ f| \leq \varepsilon \varphi_\varepsilon \circ g_n + \psi_\varepsilon \circ f + |j \circ f|.
\]
Therefore, \(0 \leq W_{\varepsilon,n} \leq \psi_\varepsilon \circ f + |j \circ f| \in L^1\). By dominated convergence,

\[
\lim_{n \to \infty} \int W_{\varepsilon,n} d\mu = 0. \quad (3)
\]

However,

\[
|j \circ f_n - j \circ g_n - j \circ f| \leq W_{\varepsilon,n} + \varepsilon \varphi_\varepsilon \circ g_n
\]

and thus

\[
I_n := \int |j \circ f_n - j \circ g_n - j \circ f| d\mu \leq \int [W_{\varepsilon,n} + \varepsilon \varphi_\varepsilon \circ g_n] d\mu.
\]

Consequently, \(\limsup I_n \leq \varepsilon C\). Now let \(\varepsilon \to 0\). □

**Remark 1.1.** (i) The conditions (3) and (4) mean that the sequence \(|j \circ f_n - j \circ g_n|\) lies eventually in the set \([-|j \circ f|, |j \circ f|] + \frac{\varepsilon n}{2} B_{1,1}\), where \(B_{1,1}\) is the unit ball of \(L^1\). In other words, the sequence \(j \circ f_n - j \circ g_n\) is almost order bounded.

(ii) The superposition operator \(J_j : L^0 \to L^0\), \(J_j(f) := j \circ f\) induced by the mapping \(j\) in the proof above can be replaced by a mapping \(J : L^0 \to L^0\) satisfying some reasonably mild conditions for keeping the statement of the Brezis–Lieb lemma.

(iii) Theorem 1.1 is equivalent to its partial case when the \(C\)-valued functions are replaced by \(\mathbb{R}\)-valued ones.

The following proposition is motivated directly by the proof of [2, Theorem 2].

**Proposition 1.2** (Brezis–Lieb lemma for mappings on \(L^0\)). Let \((\Omega, \Sigma, \mu)\) be a measure space, \(f_n = f + g_n\) be a sequence in \(L^0\) such that \(g_n \overset{a.e.}{\to} 0\), and \(J : L^0 \to L^0\) be a mapping satisfying \(J(0) = 0\) and such that the sequence \(J(f_n) - J(g_n)\) is almost order bounded. Then

\[
\lim_{n \to \infty} \int (J(f + g_n) - (J(g_n) + J(f))) d\mu = 0. \quad (5)
\]

\(< As in the proof of the Brezis–Lieb lemma above, denote \(I_n := \int |J(f + g_n) - (J(f) + J(g_n))| d\mu\). By the conditions, the sequence

\[ J(f + g_n) - (J(f) + J(g_n)) = (J(f_n) - J(g_n)) - J(f) \]

a.e.-converges to 0 and is almost order bounded. Therefore, by the generalized dominated convergence, \(\lim I_n = 0\). □

Since almost order boundedness is equivalent to uniform integrability in finite measure spaces, the following corollary is immediate.

**Proposition 1.3** (Brezis–Lieb lemma for uniform integrable sequence \(J(f_n) - J(g_n)\)). Let \((\Omega, \Sigma, \mu)\) be a finite measure space, \(f_n = f + g_n\) be a sequence in \(L^0\) such that \(g_n \overset{a.e.}{\to} 0\), and \(J : L^0 \to L^0\) be a mapping satisfying \(J(0) = 0\) and such that the sequence \(J(f_n) - J(g_n)\) is uniformly integrable. Then

\[
\lim_{n \to \infty} \int (J(f + g_n) - (J(g_n) + J(f))) d\mu = 0. \quad (6)
\]
2. Two variants of the Brezis–Lieb lemma in Riesz spaces

Recall that a sequence \( x_n \) in a Riesz space \( E \) is order convergent (or \( o \)-convergent, for short) to \( x \in E \) if there is a sequence \( z_n \in E \) satisfying \( z_n \downarrow 0 \) and \( |x_n - x| \leq z_n \) for all \( n \in \mathbb{N} \) (we write \( x_n \overset{o}{\to} x \)). In a Riesz space \( E \), a sequence \( x_n \) is unbounded order convergent (or \( uo \)-convergent, for short) to \( x \in E \) if \( |x_n - x| \wedge y \overset{o}{\to} 0 \) for all \( y \in E^+ \) (we write \( x_n \overset{uo}{\to} x \)).

Here we give two variants of the Brezis–Lieb lemma in Riesz space setting by replacing \( a.e. \)-convergence by \( uo \)-convergence, integral functionals by strictly positive functionals and the continuity of the scalar function \( j \) (in Theorem 1.1) by the so called \( \sigma \)-unbounded order continuity of the mapping \( J : E \to F \) between Riesz spaces \( E \) and \( F \). As standard references for basic notions on Riesz spaces we adopt the books [1, 7, 8] and on unbounded order continuity of mappings between Riesz spaces is used. A mapping \( f : E \to F \) between Riesz spaces is said to be \( \sigma \)-unbounded order continuous (in short, \( \sigma uo \)-continuous) if \( x_n \overset{uo}{\to} x \) in \( E \) implies \( f(x_n) \overset{uo}{\to} f(x) \) in \( F \). Clearly this definition is parallel to the well-known notion of \( \sigma \)-order continuous mappings between Riesz spaces.

Let \( F \) be a Riesz space and \( l \) be a strictly positive linear functional on \( F \). Define the following norm on \( F \):

\[
\| x \|_l := l(|x|).
\] (7)

Recall that a Banach lattice \( E \) is said to be order continuous if every order null net is norm null, and a subset \( A \) of \( E \) is said to be almost order bounded if for any \( \varepsilon > 0 \) there exists \( u_\varepsilon \in E^+ \) such that \( A \subset [u_\varepsilon, u_\varepsilon] + \varepsilon B_E \), where \( B_E \) is the closed unit ball in \( E \). We say that a net \( x_\alpha \) is almost order bounded if the set of its members is almost order bounded.

The next lemma will be used to prove a version of Brezis–Lieb lemma for arbitrary strictly positive linear functionals.

**Lemma 2.1** (See [5, Proposition 3.7]). Let \( X \) be an order continuous Banach lattice. If a net \( x_\alpha \) is almost order bounded and \( uo \)-convergent to \( x \), then \( x_\alpha \) converges to \( x \) in norm.

Suppose that \( F \) is a Riesz space and \( l \) is a strictly positive linear functional on \( F \), then the \( \| \cdot \|_l \)-completion \((F_1, \| \cdot \|_l)\) of \((F, \| \cdot \|_l)\) is an \( AL \)-space, and so it is order continuous Banach lattice. The following result is a measure-free version of Proposition 1.2.

**Proposition 2.2** (A Brezis–Lieb lemma for strictly positive linear functionals). Let \( E \) be a Riesz space and \( F_1 \) be the \( AL \)-space constructed above. Let \( J : E \to F_1 \) be \( \sigma uo \)-continuous with \( J(0) = 0 \), and \( x_n \) be a sequence in \( E \) such that:

(i) \( x_n \overset{uo}{\to} x \) in \( E \);

(ii) the sequence \((J(x_n) - J(x_n - x))_n\) is almost order bounded in \( F_1 \).

Then

\[
\lim_{n \to \infty} \| J(x_n) - J(x_n - x) - J(x) \|_l = 0.
\] (8)

\(< \) Since \( x_n \overset{uo}{\to} x \) and \( J \) is \( \sigma uo \)-continuous, then \( J(x_n) \overset{uo}{\to} J(x) \) and \( J(x_n - x) \overset{uo}{\to} J(0) = 0 \). Thus, \( J(x_n) - J(x_n - x) \overset{uo}{\to} J(x) \). It follows from Lemma 2.1 that \( \lim_{n \to \infty} \| J(x_n) - J(x_n - x) - J(x) \|_l = 0 \). \( > \)

In the following Brezis–Lieb type lemma, the \( \sigma uo \)-continuity of mappings between Riesz spaces is used.
Proposition 2.3 (A Brezis–Lieb lemma for $\sigma$uo-continuous linear functionals). Let $E$, $F$ be Riesz spaces, $l$ a $\sigma$uo-continuous linear functional on $F$, $J : E \to F$ a $\sigma$uo-continuous mapping with $J(0) = 0$, and $x_n \xrightarrow{\text{uo}} x$ in $E$. Then
\[
\lim_{n \to \infty} l(J(x_n) - J(x_n - x) - J(x)) = 0. \tag{9}
\]
Since $x_n \xrightarrow{\text{uo}} x$ and $J$ is $\sigma$uo-continuous, then $J(x_n) \xrightarrow{\text{uo}} J(x)$ and $J(x_n - x) \xrightarrow{\text{uo}} J(0) = 0$. Thus, $(J(x_n) - J(x_n - x) - J(x)) \xrightarrow{\text{uo}} 0$. But $l$ is $\sigma$uo-continuous, so $l(J(x_n) - J(x_n - x) - J(x)) \xrightarrow{\text{uo}} 0$. Since in $\mathbb{R}$ the uo-convergence, the o-convergence, and the standard convergence are all equivalent, then $\lim_{n \to \infty} l(J(x_n) - J(x_n - x) - J(x)) = 0$. $\triangleright$

Note that in opposite to Proposition 2.3, in Proposition 2.2 we do not suppose the functional $l$ to be $\sigma$uo-continuous.

References


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