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A NOTE ON SURJECTIVE POLYNOMIAL OPERATORS[1]

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A linear Markov chain is a discrete time stochastic process whose transitions depend only on the current state of the process. A nonlinear Markov chain is a discrete time stochastic process whose transitions may depend on both the current state and the current distribution of the process. These processes arise naturally in the study of the limit behavior of a large number of weakly interacting Markov processes. The nonlinear Markov processes were introduced by McKean and have been extensively studied in the context of nonlinear Chapman–Kolmogorov equations as well as nonlinear Fokker–Planck equations. The nonlinear Markov chain over a finite state space can be identified by a continuous mapping (a nonlinear Markov operator) defined on a set of all probability distributions (which is a simplex) of the finite state space and by a family of transition matrices depending on occupation probability distributions of states. Particularly, a linear Markov operator is a linear operator associated with a square stochastic matrix. It is well-known that a linear Markov operator is a surjection of the simplex if and only if it is a bijection. The similar problem was open for a nonlinear Markov operator associated with a stochastic hyper-matrix. We solve it in this paper. Namely, we show that a nonlinear Markov operator associated with a stochastic hyper-matrix is a surjection of the simplex if and only if it is a permutation of the Lotka–Volterra operator.

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1. Introduction

Let $\mathbf{I}_m := \{1, \cdots, m\}$ be a finite set, $\overline{\alpha} := \mathbf{I}_m \setminus \alpha$ be a complement of a subset $\alpha \subset \mathbf{I}_m$, and $|\alpha|$ be the number of its elements. Suppose that $\mathbb{R}^m$ is equipped with the l_1-norm $\|\mathbf{x}\|_1 := \sum_{k=1}^m |x_k|$ where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ and $\{\mathbf{e}_i\}_{i\in\mathbf{I}_m}$ stands for the standard basis. We say that $\mathbf{x} \geqslant 0$ (respectively $\mathbf{x} > 0$) if $x_i \geqslant 0$ (respectively $x_i > 0$) for all $i \in \mathbf{I}_m$. Let $\mathbf{S}^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 = 1, \mathbf{x} \geqslant 0\}$ be the $(m-1)$-dimensional standard simplex. An element of the simplex $\mathbf{S}^{m-1}$ is called *a stochastic vector*. For a stochastic vector $\mathbf{x} \in \mathbf{S}^{m-1}$, we set $\operatorname{supp}(\mathbf{x}) = \{i \in \mathbf{I}_m : x_i > 0\}, \operatorname{null}(\mathbf{x}) = \{i \in \mathbf{I}_m : x_i = 0\}$. We define a face $\Gamma_\alpha = \operatorname{conv}\{\mathbf{e}_i\}_{i\in\alpha}$ of the simplex $\mathbf{S}^{m-1}$ where $\alpha \subset \mathbf{I}_m$ and $\operatorname{conv}(\mathbf{A})$ is the convex hull of a set $\mathbf{A}$. Let $\operatorname{int}\Gamma_\alpha = \{\mathbf{x} \in \Gamma_\alpha : \operatorname{supp}(\mathbf{x}) = \alpha\}$ and $\partial\Gamma_\alpha = \Gamma_\alpha \setminus \operatorname{int}\Gamma_\alpha$ be respectively the relative interior and boundary of the face Γ_α.

Recall that a square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is called *non-negative*, written $\mathbb{P} \geqslant 0$, if $\mathbf{p}_{i\bullet} \geqslant 0$ for all $i \in \mathbf{I}_m$. A square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is called *stochastic* if each row $\mathbf{p}_{i\bullet} = (p_{i1}, \ldots, p_{im})$ is a stochastic vector for all $i \in \mathbf{I}_m$. Let $\mathscr{L} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ be a linear operator (a Markov operator) associated with a square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$, i. e.,

$$\mathscr{L}(\mathbf{x}) = \mathbf{x}\mathbb{P} = \sum_{i=1}^m x_i \mathbf{p}_{i\bullet}.$$

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It is easy to see that the linear operator $\mathscr{L} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ is a surjection if and only if it is a bijection. Indeed, the straightforward calculation shows that if $\mathscr{L} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ is a surjection then for each i there exists j such that $\mathscr{L}^{-1}(\mathbf{e}_i) = \mathbf{e}_j$ where $\mathscr{L}^{-1}(\mathbf{e}_i)$ is a preimage of the vertex $\mathbf{e}_i$ of the simplex $\mathbf{S}^{m-1}$. Consequently, surjective linear operators of the simplex are only permutation operators.

Recently, the similar problem for a quadratic operator (a nonlinear Markov operator [6]) associated with a cubic stochastic matrix was solved in the paper [5]. In general, the convexity of the quadratic operators is strongly tied up with the nonlinear optimization problems [1, 2, 4, 7] and is not an easy problem [8]. In this paper, we provide a criterion for surjectivity of polynomial operators associated with stochastic hyper-matrices.

2. Polynomial Operators Associated with Stochastic Hyper-Matrices

Let $\mathscr{P} = (p_{i_1\dots i_k})_{i_1,\dots,i_k=1}^m$ be a k-order m-dimensional hyper-matrix. We define the following vectors and matrices

$$\mathbf{p}_{i_1\dots i_{k-1}\bullet} = \big(p_{i_1\dots i_{k-1}1}, \dots, p_{i_1\dots i_{k-1}m}\big), \quad \mathbb{P}_{i_1\dots i_{k-2}\bullet\bullet} = \big(p_{i_1\dots i_{k-2}jl}\big)_{j,l=1}^m,$$

for any $i_1, \dots, i_{k-1} \in \mathbf{I}_m$. In what follows, we denote $i_{[1:l]} := i_1 \dots i_l$ for index.

A hyper-matrix $\mathscr{P} = (p_{i_1\dots i_k})_{i_1,\dots,i_k=1}^m$ is called *non-negative* and written $\mathscr{P} \geqslant 0$ if $\mathbf{p}_{i_{[1:k-1]}\bullet} \geqslant 0$ for all $i_1, \dots, i_{k-1} \in \mathbf{I}_m$. A hyper-matrix $\mathscr{P} = (p_{i_1\dots i_k})_{i_1,\dots,i_k=1}^m$ is called *stochastic* if each vector $\mathbf{p}_{i_{[1:k-1]}\bullet}$ is stochastic for all $i_1, \dots, i_{k-1} \in \mathbf{I}_m$.

We define a polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ associated with k-order m-dimensional stochastic hyper-matrix $\mathscr{P} = (p_{i_1\dots i_k})_{i_1,\dots,i_k=1}^m$ as follows

$$\mathfrak{P}(\mathbf{x}) = \sum_{i_1=1}^m \cdots \sum_{i_{k-1}=1}^m x_{i_1} \dots x_{i_{k-1}} \mathbf{p}_{i_{[1:k-1]}\bullet} \tag{1}$$

for any $\mathbf{x} \in \mathbf{S}^{m-1}$. It is easy to check that

$$\mathfrak{P}(\mathbf{x}) = \mathbf{x}\mathbb{P}_{\mathbf{x}} \tag{2}$$

where

$$\mathbb{P}_{\mathbf{x}} = \sum_{i_1=1}^m \cdots \sum_{i_{k-2}=1}^m x_{i_1} \dots x_{i_{k-2}} \mathbb{P}_{i_{[1:k-2]}\bullet\bullet} = \big(p_{jl}(\mathbf{x})\big)_{j,l=1}^m$$

is a square stochastic matrix for any $\mathbf{x} \in \mathbf{S}^{m-1}$. Due to the matrix form (2), the polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ associated with k-order m-dimensional stochastic hyper-matrix $\mathscr{P}$ is *a nonlinear Markov operator* (see [6]). Unlike the classical Markov chain, the nonlinear Markov chain is a stochastic process whose transition matrix $\mathbb{P}_{\mathbf{x}}$ may depend not only on *the current state* of the process but also on *the current distribution* $\mathbf{x}$ of the process (see [3]).

Throughout this paper, without loss of generality, we assume that

$$\mathbf{p}_{i_1\dots i_{k-1}\bullet} = \mathbf{p}_{i_{\pi(1)}\dots i_{\pi(k-1)}\bullet}$$

for any $i_1, \dots, i_{k-1} \in \mathbf{I}_m$ and any permutation π of the set $\mathbf{I}_{k-1}$. We also assume that $m \geqslant k$. We need the following auxiliary results.

Proposition 2.1 [6]**.** *The following statements hold:*

(i) $\operatorname{supp}(\mathfrak{P}(\mathbf{x})) = \bigcup_{i_{[1:k-1]} \in \operatorname{supp}(\mathbf{x})} \operatorname{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet})$;

(ii) $\mathrm{null}(\mathfrak{P}(\mathbf{x})) = \bigcap_{i_{[1:k-1]}\in\mathrm{supp}(\mathbf{x})} \mathrm{null}(\mathbf{p}_{i_{[1:k-1]}\bullet})$;

(iii) $\mathfrak{P}(\mathrm{int}\,\Gamma_\alpha) \subset \mathrm{int}\,\Gamma_\beta$ *where* $\beta = \bigcup_{i_{[1:k-1]}\in\alpha} \mathrm{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet})$;

(iv) $\mathfrak{P}(\mathrm{int}\,\Gamma_\alpha) \subset \mathrm{int}\,\Gamma_\beta$ *if and only if* $\mathfrak{P}(\mathbf{x}^{(0)}) \in \mathrm{int}\,\Gamma_\beta$ *for some* $\mathbf{x}^{(0)} \in \mathrm{int}\,\Gamma_\alpha$.

An absorbing state played an important role in the theory of the classical Markov chains. Analogously, the concept of absorbing sets for nonlinear Markov chains was introduced in the paper [6].

Definition 2.1 [6]. A subset $\alpha \subset \mathbf{I}_m$ is called absorbing if one has that

$$\overline{\alpha} = \bigcap_{i_{[1:k-1]}\in\alpha} \mathrm{null}(\mathbf{p}_{i_{[1:k-1]}\bullet}).$$

It is clear that $\alpha \subset \mathbf{I}_m$ is an absorbing set if and only if

$$\alpha = \bigcup_{i_{[1:k-1]}\in\alpha} \mathrm{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}).$$

The following result presents an insight of an absorbing set.

Proposition 2.2 [6]. *The following statements are equivalent:*
(i) *A subset* $\alpha \subset \mathbf{I}_m$ *is absorbing;*
(ii) *One has that* $\mathfrak{P}(\mathrm{int}\,\Gamma_\alpha) \subset \mathrm{int}\,\Gamma_\alpha$;
(iii) *One has that* $\mathfrak{P}\left(\mathbf{x}^{(0)}\right) \in \mathrm{int}\,\Gamma_\alpha$ *for some* $\mathbf{x}^{(0)} \in \mathrm{int}\,\Gamma_\alpha$.

Proposition 2.3. *If any subset* $\alpha \subset \mathbf{I}_m$ *with* $|\alpha| \leqslant k-1$ *is absorbing then so are all subsets of* $\mathbf{I}_m$.

$\lhd$ Suppose that any subset $\alpha \subset \mathbf{I}_m$ with $|\alpha| \leqslant k-1$ is absorbing. It means that $\mathrm{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) \subset \alpha$ for any $i_1, \cdots, i_{k-1} \in \alpha$. In particular, the sets $\alpha^\circ = \{i_1^\circ, \cdots, i_{k-1}^\circ\}$ and $\beta^\circ = \{j^\circ\}$ are absorbing for the given indices $i_1^\circ, \cdots, i_{k-1}^\circ, j^\circ \in \mathbf{I}_m$ (the repetition of indices is allowed). We then obtain that $\mathrm{supp}(\mathbf{p}_{j^\circ\cdots j^\circ\bullet}) = \{j^\circ\}$ and $\mathrm{supp}(\mathbf{p}_{i_1^\circ\cdots i_{k-1}^\circ\bullet}) \subset \{i_1^\circ, \cdots i_{k-1}^\circ\} = \alpha^\circ$ for any given indices $i_1^\circ, \cdots, i_{k-1}^\circ, j^\circ \in \mathbf{I}_m$ (the repetition of indeces is allowed). Hence, for any $\beta \subset \mathbf{I}_m$ one has that

$$\bigcup_{i_{[1:k-1]}\in\beta} \mathrm{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) = \bigcup_{j\in\beta} \mathrm{supp}(\mathbf{p}_{j\cdots j\bullet}) \cup \bigcup_{i_\nu\neq i_\mu} \mathrm{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) = \beta.$$

It means that β is an absorbing subset. This completes the proof. $\rhd$

Lemma 2.1. *If any subset* $\alpha \subset \mathbf{I}_m$ *with* $|\alpha| \leqslant k-1$ *is absorbing then the polynomial operator* $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ *is a surjection.*

$\lhd$ Due to Propositions 2.2 and 2.3, the polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ maps each face of the simplex $\mathbf{S}^{m-1}$ into itself. It is well-known in algebraic topology that any continuous mapping which maps each face of the simplex $\mathbf{S}^{m-1}$ into itself is a surjection of the simplex $\mathbf{S}^{m-1}$ (see Lemma 1, [5]). This completes the proof. $\rhd$

3. Surjective Polynomial Operators vs Lotka–Volterra Operators

We recall a definition of Lotka–Volterra operators (see [5]).

Definition 3.1. A polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ is called *the Lotka–Volterra operator* if $\mathrm{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) \subset \{i_1, \ldots, i_{k-1}\}$ for any $i_1, \ldots, i_{k-1} \in \mathbf{I}_m$.

We provide a criterion for the Lotka–Volterra operator in terms of absorbing sets.

Lemma 3.1. *The following statements are equivalent:*
(i) The polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ *is the Lotka–Volterra operator;*
(ii) Any subset $\alpha \subset \mathbf{I}_m$ *with* $|\alpha| \leqslant k-1$ *is absorbing;*
(iii) One has that $\mathfrak{P}^{-1}(\text{int } \Gamma_\alpha) = \text{int } \Gamma_\alpha$ *for any subset* $\alpha \subset \mathbf{I}_m$ *with* $|\alpha| \leqslant k-1$.

Remark 3.1. We always assume $\text{int}\,\Gamma_\alpha := \Gamma_\alpha$ for the subset $\alpha \subset \mathbf{I}_m$ with $|\alpha| = 1$.

$\triangleleft$ We prove the following implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Let $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ be the Lotka–Volterra operator. We then have that $\text{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) \subset \{i_1, \ldots, i_{k-1}\}$ for any $i_1, \ldots, i_{k-1} \in \mathbf{I}_m$ (the repetition of indices is allowed). Particularly, $\text{supp}(\mathbf{p}_{j\cdots j\bullet}) = \{j\}$ for any $j \in \mathbf{I}_m$. Hence, for any $\alpha \subset \mathbf{I}_m$ with $|\alpha| \leqslant k-1$ one has that

$$\bigcup_{i_{[1:k-1]}\in\alpha} \text{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) = \bigcup_{j\in\alpha} \text{supp}(\mathbf{p}_{j\ldots j\bullet}) \cup \bigcup_{i_\nu \neq i_\mu} \text{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) = \alpha.$$

It means that α is an absorbing subset.

$(ii) \Rightarrow (iii)$: Suppose that any subset $\alpha \subset \mathbf{I}_m$ with $|\alpha| \leqslant k-1$ is absorbing. We then obtain that $\text{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) \subset \alpha$ for any $i_1, \ldots, i_{k-1} \in \alpha$. Particularly, since the subset $\alpha^\circ = \{j\}$ is absorbing, we have that $\text{supp}(\mathbf{p}_{j\ldots j\bullet}) = \{j\}$ for any $j \in \mathbf{I}_m$. It follows from Proposition 2.2, (ii) that $\mathfrak{P}(\text{int}\,\Gamma_\alpha) \subset \text{int}\,\Gamma_\alpha$ for any absorbing subset $\alpha \subset \mathbf{I}_m$. Moreover, if $\mathfrak{P}^{-1}(\text{int } \Gamma_\alpha) \setminus \text{int } \Gamma_\alpha \neq \varnothing$ then there exists $\mathbf{y} \in \mathbf{S}^{m-1}$ with $\beta := \text{supp}(\mathbf{y})$ such that $\beta \setminus \alpha \neq \varnothing$ and $\mathfrak{P}(\mathbf{y}) \in \text{int}\,\Gamma_\alpha$. Then it follows from Proposition 2.1, (iv) that $\mathfrak{P}(\text{int}\,\Gamma_\beta) \subset \text{int } \Gamma_\alpha$. Since $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ is continuous, we have that $\mathfrak{P}(\Gamma_\beta) = \overline{\mathfrak{P}(\text{int } \Gamma_\beta)} \subset \overline{\text{int}\,\Gamma_\alpha} = \Gamma_\alpha$. Particularly, $\mathfrak{P}(\mathbf{e}_j) = \mathbf{p}_{j\ldots j\bullet} \in \Gamma_\alpha$ (or equivalently $\text{supp}(\mathbf{p}_{j\cdots j\bullet}) \subset \alpha$) for $j \in \beta \setminus \alpha$. However, this contradicts to the fact that the singleton $\{j\}$, $j \in \beta \setminus \alpha$ is an absorbing set (or equivalently $\text{supp}(\mathbf{p}_{j\cdots j\bullet}) = \{j\}$). Therefore, we have that $\mathfrak{P}^{-1}(\text{int}\,\Gamma_\alpha) = \text{int}\,\Gamma_\alpha$.

$(iii) \Rightarrow (i)$: Suppose that $\mathfrak{P}^{-1}(\text{int } \Gamma_\alpha) = \text{int}\,\Gamma_\alpha$ for any subset $\alpha \subset \mathbf{I}_m$ with $|\alpha| \leqslant k-1$. We then obtain from Proposition 2.1, (iii) that

$$\bigcup_{i_{[1:k-1]}\in\alpha} \text{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) = \alpha.$$

Particularly, we get that $\text{supp}(\mathbf{p}_{i_{[1:k-1]}\bullet}) \subset \alpha$ for any $i_1, \ldots, i_{k-1} \in \alpha$. Let us now fix indices $i_1^\circ, \ldots, i_{k-1}^\circ \in \mathbf{I}_m$ (the repetition of indices is allowed). Then, for the set $\alpha^\circ = \{i_1^\circ, \ldots, i_{k-1}^\circ\}$ we have that $\text{supp}(\mathbf{p}_{i_1^\circ\cdots i_{k-1}^\circ\bullet}) \subset \{i_1^\circ, \ldots, i_{k-1}^\circ\} = \alpha^\circ$. Since the indices $i_1^\circ, \ldots, i_{k-1}^\circ \in \mathbf{I}_m$ are arbitrary chosen, the last inclusion means that $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ is the Lotka–Volterra operator. This completes the proof. $\triangleright$

We are now ready to formulate the main result of the paper.

Theorem 3.1. *Let* $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ *be a polynomial operator. Then the following statements are equivalent:*
(i) The polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ *is a surjection;*
(ii) There exists a permutation π *of the set* $\mathbf{I}_m$ *such that for any* $1 \leqslant l \leqslant k-1$ *and for any* $i_1, \ldots, i_l \in \mathbf{I}_m$ *one has that* $\mathfrak{P}^{-1}(\text{int } \Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}}) = \text{int } \Gamma_{\mathbf{e}_{\pi(i_1)}\cdots\mathbf{e}_{\pi(i_l)}}$ *where* $\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}} = \text{conv}\{\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_l}\}$;
(iii) There exists a permutation matrix Π *such that* $\Pi \circ \mathfrak{P}$ *is the Lotka–Volterra operator.*

Remark 3.2. We always assume that $\text{int}\,\Gamma_{\mathbf{e}_i} := \{\mathbf{e}_i\}$ for any $i \in \mathbf{I}_m$.

$\triangleleft$ We prove the following implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Suppose that the polynomial operator $\mathfrak{P} : \mathbf{S}^{m-1} \to \mathbf{S}^{m-1}$ is a surjection. Let $\mathfrak{P}^{-1}(\mathbf{e}_j)$ be a preimage (which is nonempty) of the vertex $\mathbf{e}_j$ for $j \in \mathbf{I}_m$. Obviously, if $\mathbf{x} \in \mathfrak{P}^{-1}(\mathbf{e}_j)$ with $\operatorname{supp}(\mathbf{x}) = \alpha$ then $\Gamma_\alpha \subset \mathfrak{P}^{-1}(\mathbf{e}_j)$ (see Proposition 2.1, (iv)). Hence, $\mathfrak{P}^{-1}(\mathbf{e}_j)$ is a face or a union of faces of the simplex $\mathbf{S}^{m-1}$ for any $j \in \mathbf{I}_m$. Consequently, the set $\{\mathfrak{P}^{-1}(\mathbf{e}_1), \cdots, \mathfrak{P}^{-1}(\mathbf{e}_m)\}$ consists of (at least) m mutually disjoint faces of the simplex $\mathbf{S}^{m-1}$. This is possible if and only if there exists a permutation π of the set $\mathbf{I}_m$ such that $\mathfrak{P}^{-1}(\mathbf{e}_j) = \mathbf{e}_{\pi(j)}$ for any $j \in \mathbf{I}_m$. Let us now show $\mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}}) = \operatorname{int}\Gamma_{\mathbf{e}_{\pi(i_1)}\ldots\mathbf{e}_{\pi(i_l)}}$ for any $i_1, \ldots, i_l \in \mathbf{I}_m$ by means of mathematical induction with respect to l where $1 \leqslant l \leqslant k-1$. Obviously, if $\mathbf{y} \in \mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})$ with $\operatorname{supp}(\mathbf{y}) = \beta$ then $\operatorname{int}\Gamma_\beta \subset \mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})$ and $\Gamma_\beta \subset \mathfrak{P}^{-1}(\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})$ (see Proposition 2.1, (iv)). Moreover, if $\beta \setminus \{\pi(i_1), \ldots, \pi(i_l)\} \neq \varnothing$ (or equivalently $\pi^{-1}(\beta) \setminus \{i_1, \ldots, i_l\} \neq \varnothing$) then $\mathbf{e}_{\pi(j)} \in \mathfrak{P}^{-1}(\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})$ for some $j \in \pi^{-1}(\beta) \setminus \{i_1, \ldots, i_l\}$. However, it contradicts to $\mathbf{e}_{\pi(j)} = \mathfrak{P}^{-1}(\mathbf{e}_j)$. Therefore, we must have that $\beta \subset \{\pi(i_1), \ldots, \pi(i_l)\}$. On the other hand, due to mathematical induction, we also have that $\{\pi(i_1), \ldots, \pi(i_l)\} \setminus \beta = \varnothing$. Hence, we get that $\beta = \{\pi(i_1), \ldots, \pi(i_l)\}$. Since the point $\mathbf{y} \in \mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})$ is arbitrary chosen, we obtain that $\mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}}) \subset \operatorname{int}\Gamma_{\mathbf{e}_{\pi(i_1)}\ldots\mathbf{e}_{\pi(i_l)}}$. The inclusion $\operatorname{int}\Gamma_{\mathbf{e}_{\pi(i_1)}\ldots\mathbf{e}_{\pi(i_l)}} \subset \mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})$ follows from Proposition 2.1, (iv). Consequently, $\mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}}) = \operatorname{int}\Gamma_{\mathbf{e}_{\pi(i_1)}\ldots\mathbf{e}_{\pi(i_l)}}$ for any $i_1, \ldots, i_l \in \mathbf{I}_m$ and $1 \leqslant l \leqslant k-1$.

$(ii) \Rightarrow (iii)$: Suppose that there exists a permutation π such that $\mathfrak{P}^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}}) = \operatorname{int}\Gamma_{\mathbf{e}_{\pi(i_1)}\ldots\mathbf{e}_{\pi(i_l)}}$ for any $i_1, \ldots, i_l \in \mathbf{I}_m$ and $1 \leqslant l \leqslant k-1$. Particularly, we have that $\mathfrak{P}^{-1}(\mathbf{e}_j) = \mathbf{e}_{\pi(j)}$ for any $j \in \mathbf{I}_m$. We now define a permutation matrix Π (associated with the permutation π) as follows $\Pi(\mathbf{e}_j) := \mathbf{e}_{\pi(j)}$ for any $j \in \mathbf{I}_m$. Obviously, we obtain that

$$\Pi\big(\mathfrak{P}(\operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}})\big) = \operatorname{int}\Gamma_{\mathbf{e}_{i_1}\ldots\mathbf{e}_{i_l}}, \quad \forall\, i_1, \ldots, i_l \in \mathbf{I}_m,\ \forall\, 1 \leqslant l \leqslant k-1.$$

Due to Lemma 3.1, the polynomial operator $\Pi \circ \mathfrak{P}$ is the Lotka–Volterra operator.

$(iii) \Rightarrow (i)$: Suppose that there exists a permutation matrix Π such that $\mathfrak{P}_\Pi := \Pi \circ \mathfrak{P}$ is the Lotka–Volterra operator. Due to Lemmas 2.1 and 3.1, the Lotka–Volterra operator $\mathfrak{P}_\Pi$ is a surjection and so is the polynomial operator $\mathfrak{P} = \Pi^{-1} \circ \mathfrak{P}_\Pi$. This completes the proof. $\rhd$

References

1. *Barvinok A. I.* Problems of distance geometry and convex properties of quadratic maps // Discrete Comput. Geom.—1995.—Vol. 13 (2).—P. 189–202. DOI 10.1007/BF02574037.
2. *Hiriart-Urruty J.-B., Torki M.* Permanently going back and forth between the “Quadratic World” and the “Convexity World” in optimization // Appl. Math. Optim.—2002.—Vol. 45.—P. 169–184. DOI 10.1007/s00245-001-0034-6.
3. *Kolokoltsov V.* Nonlinear Markov Processes and Kinetic Equations.—Cambridge Univ., 2010. DOI 10.1017/CBO9780511760303.
4. *Polyak B. T.* Convexity of quadratic transformations and its use in Control and Optimization // J. Optim. Theory Appl.—1998.—Vol. 99.—P. 553–583. DOI 10.1023/A:1021798932766.
5. *Saburov M.* On the surjectivity of quadratic stochastic operators acting on the simplex // Math. Notes.—2016.—Vol. 99 (4).—P. 623–627. DOI 10.1134/S0001434616030391.
6. *Saburov M.* Ergodicity of nonlinear Markov operators on the finite dimensional space // Nonlinear Anal. Theory Methods.—2016.—Vol. 143.—P. 105–119. DOI 10.1016/j.na.2016.05.006.
7. *Sheriff J. L.* The convexity of quadratic maps and the controllability of coupled systems: Doctoral dissertation.—Harvard Univ., 2013.
8. *Vershik A. M.* Quadratic forms positive on a cone and quadratic duality // J. Soviet Math.—1984.—Vol. 36 (1).—P. 39–56. DOI 10.1007/BF01104972.

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ЗАМЕЧАНИЕ О СЮРЪЕКТИВНЫХ ПОЛИНОМИАЛЬНЫХ ОПЕРАТОРАХ

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Линейная цепь Маркова является случайным процессом с дискретными состояниями, переходы которого зависят только от текущего состояния процесса. Нелинейная цепь Маркова — случайный процесс с дискретными состояниями, переходы которого могут зависеть как от текущего состояния, так и текущего распределения процесса. Эти процессы естественным образом возникают при изучении предельного поведения большого количества слабо взаимодействующих марковских процессов. Нелинейные марковские процессы были введены Маккином и широко изучались в контексте нелинейных уравнений Чапмана — Колмогорова, а также нелинейных уравнений Фоккера — Планка. Нелинейная цепь Маркова над конечным пространством состояний может быть определена непрерывным отображением (нелинейным оператором Маркова), определяемым на множестве всех вероятностных распределений (являющемся симплексом) конечного пространства состояний семейством матриц перехода, зависящих от распределения вероятностей занятия состояний. В частности, линейный оператор Маркова является линейным оператором, связанным с квадратной стохастической матрицей. Хорошо известно, что линейный оператор Маркова будет сюръекцией симплекса в том и только в том случае, когда он является биекцией. Аналогичная задача для нелинейного оператора Маркова, связанного со стохастической гипер-матрицей, оставалась открытой. Она решена в данной статье, а именно, показано, что нелинейный оператор Марков, связанный со стохастической гипер-матрицей, является сюръекцией симплекса, если и только если он является перестановкой оператор Лотки — Вольтерра.

Ключевые слова: стохастическая гипер-матрица, полиномиальный оператор, оператор Лотки — Вольтерра.