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CHARACTERIZATIONS OF FINITE DIMENSIONAL  
ARCHIMEDEAN VECTOR LATTICES

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*To Professor Anatoly Kusraev  
with warmest wishes on the occasion  
of his 65th anniversary*

**Abstract.** In this paper, we give some necessary and sufficient conditions for an Archimedean vector lattice  $A$  to be of finite dimension. In this context, we give three characterizations. The first one contains the relation between the vector lattice  $A$  to be of finite dimension and its universal completion  $A^u$ . The second one shows that the vector lattice  $A$  is of finite dimension if and only if one of the following two equivalent conditions holds : (a) every maximal modular algebra ideal in  $A^u$  is relatively uniformly complete or (b)  $\text{Orth}(A, A^u) = Z(A, A^u)$  where  $\text{Orth}(A, A^u)$  and  $Z(A, A^u)$  denote the vector lattice of all orthomorphisms from  $A$  to  $A^u$  and the sublattice consisting of orthomorphisms  $\pi$  with  $|\pi(x)| \leq \lambda|x|$  ( $x \in A$ ) for some  $0 \leq \lambda \in \mathbb{R}$ , respectively. It is well-known that any universally complete vector lattice  $A$  is of the form  $C^\infty(X)$  for some Hausdorff extremally disconnected compact topological space  $X$ . The point  $x \in X$  is called  $\sigma$ -isolated if the intersection of every sequence of neighborhoods of  $x$  is a neighborhood of  $x$ . The last characterization of finite dimensional Archimedean vector lattices is the following. Let  $A$  be a vector lattice and let  $A^u (= C^\infty(X))$  be its universal completion. Then  $A$  is of finite dimension if and only if each element of  $X$  is  $\sigma$ -isolated. Bresar in [1] raised a question to find new examples of zero product determined algebras. Finally, as an application, we give a positive answer to this question.

**Key words:** hyper-Archimedean vector lattice,  $f$ -algebra, universally complete vector lattice.

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## 1. Introduction

Throughout the paper,  $\mathbb{R}$  and  $\mathbb{C}$  denote real numbers and complex numbers, respectively and let  $\mathbb{N} = \{1, 2, \dots\}$ . For a set  $A$ ,  $A^n$  denotes the cartesian product  $A \times \underbrace{\dots}_{n \text{ times}} \times A$ ,  $n \in \mathbb{N}$ .

The Gelfand–Mazur Theorem states that if  $A$  is an associative normed real division algebra, then  $A$  is isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$  or the quaternion field. For the details, we refer to [2]. For the case of lattice-ordered algebras, Huijsmans [3] proved that an Archimedean lattice-ordered algebra with unit element  $e > 0$  in which every positive element has a positive inverse is lattice and algebra isomorphic to  $\mathbb{R}$ . Uyar [4] gave an alternative proof to the result of Huijsmans for Banach lattice algebras. Later on, these two results are generalized and combined in [5] by using an easy observation as follows:

**Theorem 1.** *Let  $A$  be an Archimedean lattice-ordered algebra with unit element  $e > 0$ . Then the following statements are equivalent:*

- (i) *Every positive element has a positive inverse.*
- (ii)  *$A$  is a  $d$ -algebra and every positive element has a positive inverse.*
- (iii)  *$A$  is an  $f$ -algebra and each positive element has a positive inverse.*
- (iv)  *$A$  is an almost  $f$ -algebra and each positive element has an inverse.*
- (v)  *$A$  is an almost  $f$ -algebra and each nonzero element has an inverse.*
- (vi)  *$A$  is order and algebra isomorphic to  $\mathbb{R}$ .*

*If one of the statements above is satisfied and  $A$  is a normed lattice-ordered algebra with  $\|e\| = 1$ , then  $A$  is also isometric to  $\mathbb{R}$ .*

As far as we know, no attention at all has been paid in the literature to the problem when a lattice-ordered algebra is of finite dimension. The aim of this paper is to give a positive answer to this problem. In connection with our problem, Bresar studied the class of finite dimensional spaces which are zero product determined. Recall that an algebra  $A$  over a field  $\mathbb{K}$  is said to be *zero product determined* if for every bilinear map  $f : A \times A \rightarrow B$ , where  $B$  is an arbitrary vector space over  $\mathbb{K}$ , with the property that for all  $x, y \in A$ ,  $f(x, y) = 0$  whenever  $xy = 0$ , is of the form  $f(x, y) = \Phi(xy)$  for some linear map  $\Phi : A \rightarrow B$ . This concept was introduced in [1]. The original motivation for this concept was problems on the zero product preserving linear maps. Recently, Bresar [1] proved the following result.

**Theorem 2.** *A finite dimensional algebra is zero product determined if and only if it is generated by idempotents.*

Bresar pointed out that the main purpose of the paper [1] was to find new examples of zero product determined algebras, but ultimately it is restricted to an unexpected characterization of finite-dimensional algebras that are generated by idempotents and the initial problem of finding new examples still remains open. Moreover, the problem of finding other classes of algebras for which the characterization of Theorem 1 holds is fully open.

As an application of our study, we will give a new class of non-finite dimensional zero product algebras for which the characterization of the previous theorem holds.

## 2. Preliminaries

In order to avoid unnecessary repetitions, we assume that all vector lattices under consideration are *Archimedean*.

In the following lines, we recall some definitions and basic facts about vector lattices, lattice-ordered algebras and multilinear maps.

For the unexplained terminology on vector lattices, lattice-ordered algebras and multilinear maps, we refer the reader to [6, 7, 8, 9].

Given a vector lattice  $A$ , the set  $A^+ = \{a \in A : a \geq 0\}$  is called *the positive cone* of  $A$ . Let  $a, e \in A^+$ , then  $e$  is called a *component* of  $a$  if  $e \wedge (a - e) = 0$ .

An algebra  $A$  which is simultaneously a vector lattice such that the partial ordering and the multiplication in  $A$  are compatible, that is  $a, b \in A^+$  implies  $ab \in A^+$ , is called *lattice-ordered algebra* (briefly a  *$\ell$ -algebra*). An  $\ell$ -algebra  $A$  is called an  *$f$ -algebra* if  $A$  verifies the property that  $a \wedge b = 0$  and  $c \geq 0$  imply  $ac \wedge b = ca \wedge b = 0$ . Any  $f$ -algebra is automatically commutative and has positive squares. An  $\ell$ -algebra  $A$  is called an *almost  $f$ -algebra* whenever it follows from  $a \wedge b = 0$  that  $ab = ba = 0$ . An  $\ell$ -algebra  $A$  is called a  *$d$ -algebra* if  $A$  verifies the property that  $a \wedge b = 0$  and  $c \geq 0$  imply  $ac \wedge bc = ca \wedge cb = 0$ .

The vector lattice  $A$  is called *Dedekind complete* if for each non-empty subset  $B$  of  $A$  which is bounded above,  $\sup B$  exists in  $A$ . The vector lattice  $A$  is called *laterally complete*

if every orthogonal system in  $A$  has a supremum in  $A$ . If  $A$  is Dedekind complete and laterally complete, then  $A$  is called *universally complete*. Every vector lattice  $A$  has a *universal completion*  $A^u$ , this means that there exists a unique (up to a lattice isomorphism) universally complete vector lattice  $A^u$  such that  $A$  can be identified with an order dense sublattice of  $A^u$  (see [6, Section 8, Exercise 13] for an interesting approach to the existence of the universal completion by using orthomorphisms).

A vector lattice  $A$  is said to have *the countable sup property*, if whenever an arbitrary subset  $D$  has a supremum, then there exists an at most countable subset  $C$  of  $D$  with  $\sup C = \sup D$ . A Dedekind complete vector lattice with the countable sup property is called *super Dedekind complete* vector lattice.

Let  $A$  be a vector lattice. A subset  $S$  of  $A^+$  is called an *orthogonal system* of  $A$  if  $0 \notin S$  and  $u \wedge v = 0$  for each pair  $(u, v)$  of distinct elements in  $S$ . It follows from Zorn's lemma that every orthogonal system of  $A$  is contained in a maximal orthogonal system.

A subset  $S$  in a vector lattice  $E$  is called *solid* if it follows from  $|u| \leq |v|$  in  $E$  and  $v \in S$  that  $u \in S$ . A solid vector subspace of a vector lattice is called *an ideal*. The ideal  $P$  in a vector lattice is *prime* whenever it follows from  $\inf(a, b) \in P$  that at least one of  $a \in P$  or  $b \in P$  holds. A *principal ideal* of a vector lattice  $E$  is any ideal generated by a singleton  $\{u\}$  denoted by  $E_u$ . Clearly,  $E_u = \{v \in E : \exists \lambda \geq 0 \text{ such that } |v| \leq \lambda|u|\}$ .

An order closed ideal in a vector lattice is called *a band*. A band  $B$  of a vector lattice  $E$  is said to be a *projection band* if  $B \oplus B^d = E$  where  $B^d$  denotes *the disjoint complement* of  $B$ . A vector lattice has *the projection property* if every band is a projection band.

Let  $A$  be a vector lattice and  $v \in A^+$ . Then the sequence  $(a_n)_{n \in \mathbb{N}}$  in  $A$  is called *(v) relatively uniformly convergent* to  $a \in A$  if for every real number  $\varepsilon > 0$ , there exists a natural number  $n_\varepsilon$  such that  $|a_n - a| \leq \varepsilon v$  for all  $n \geq n_\varepsilon$ . This will be denoted by  $a_n \rightarrow a (v)$ . If  $a_n \rightarrow a (v)$  for some  $0 \leq v \in A$ , then the sequence  $(a_n)_{n \in \mathbb{N}}$  is called *(relatively) uniformly convergent* to  $a$ , which will be denoted by  $a_n \rightarrow a (r.u)$ . The notion of *(v) (relatively) uniformly Cauchy* sequence is defined in the obvious way. A vector lattice is called *(relatively) uniformly complete* if every relatively uniformly Cauchy sequence in  $A$  has a unique limit. Relatively uniformly limits are unique if  $A$  is Archimedean.

Let  $A$  and  $B$  be vector lattices. A multilinear map  $\Psi : A^n \rightarrow B$  is said to be *positive* whenever  $(a_1, \dots, a_n) \in (A^+)^n$  implies  $\Psi(a_1, \dots, a_n) \in B^+$ . A multilinear map  $\Psi$  is said to be *orthosymmetric* if for all  $(a_1, \dots, a_n) \in A^n$  such that  $a_i \wedge a_j = 0$  for some  $1 \leq i, j \leq n$  implies  $\Psi(a_1, \dots, a_n) = 0$ .

### 3. Main results

We start with some auxiliary results which will be used in the sequel.

**Proposition 1.** *Let  $A$  be a vector lattice and  $n \in \mathbb{N}$ . Then the followings are equivalent:*

- 1)  $A = I_1 \oplus I_2 \oplus \dots \oplus I_n$  for some simple order ideals  $I_1, I_2, \dots, I_n$  in  $A$ .
- 2)  $A = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \dots \oplus \mathbb{R}x_n$  for some elements  $x_1, x_2, \dots, x_n \in A$ .

$\triangleleft$  (1)  $\Rightarrow$  (2) Since  $I_i$  is simple for each  $i = 1, 2, \dots, n$ , and according to [3, Proposition 1], it follows that  $I_i = \mathbb{R}x_i$  for some elements  $x_1, x_2, \dots, x_n \in A$ . Consequently,  $A = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \dots \oplus \mathbb{R}x_n$  for some  $n \in \mathbb{N}$  and some elements  $x_1, x_2, \dots, x_n \in A$ .

(2)  $\Rightarrow$  (1) This direction is trivial.  $\triangleright$

**DEFINITION 1.** The depth of a vector lattice is the supremum (possibly infinite) of the lengths of maximal orthogonal system.

**Proposition 2.** *Let  $A$  be a vector lattice. Then the followings are equivalent:*

- 1)  $A$  has the projection property and its depth is finite;
- 2)  $A = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \dots \oplus \mathbb{R}x_n$  for some  $n \in \mathbb{N}$  and some elements  $x_1, x_2, \dots, x_n \in A$ .

$\triangleleft$  (1)  $\Rightarrow$  (2): Let  $n \in \mathbb{N}$  be the depth of  $A$  and  $\{e_1, e_2, \dots, e_n\}$  be the maximal orthogonal system of length  $n$  in  $A$ . Since  $A$  has the projection property,  $A = \{e_1\}^{dd} \oplus \{e_2\}^{dd} \oplus \dots \oplus \{e_n\}^{dd} \oplus \dots \oplus \{e_n\}^{dd}$ . Let  $1 \leq i \leq n$  be fixed and  $y \in \{e_i\}^{dd}$ . Then  $y = y^+ - y^-$  where  $y^+, y^- \in \{e_i\}^{dd}$ . Let  $f_1 = \pi_{\{y^+\}^{dd}}(e_i)$  and  $f_2 = \pi_{\{y^+\}^d}(e_i)$  where  $\pi_{\{y^+\}^{dd}}$  and  $\pi_{\{y^+\}^d}$  are the band projections on  $\{e_1, e_2, \dots, e_n\}$  with ranges  $\{y^+\}^{dd}$  and  $\{y^+\}^d$ , respectively. It is easy to see that the system  $\{e_1, e_2, \dots, e_{i-1}, f_1, f_2, e_{i+1}, \dots, e_n\}$  is an orthogonal system of length  $n + 1$ . Therefore, either  $f_1 = 0$  or  $f_2 = 0$ .

CASE 1: If  $f_1 = 0$ , then  $e_i = f_2 \in \{y^+\}^d$ . Hence  $\{y^+\}^{dd} \subset \{e_i\}^{dd} \subset \{y^+\}^d$  and so  $y^+ = 0$ .

CASE 2: If  $f_2 = 0$ , then  $e_i = f_1 \in \{y^+\}^{dd}$ . Hence  $\{y^-\}^{dd} \subset \{e_i\}^{dd} \subset \{y^+\}^{dd} \subset \{y^-\}^d$  and so  $y^- = 0$ .

Then, the band  $\{e_i\}^{dd}$  is totally ordered for all  $1 \leq i \leq n$ . By [3, Proposition 1],  $\{e_i\}^{dd} = \mathbb{R}e_i$  for all  $1 \leq i \leq n$ . Then  $A = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \dots \oplus \mathbb{R}e_n$  for some  $n \in \mathbb{N}$ .

The implication (2)  $\Rightarrow$  (1) is trivial.  $\triangleright$

To clarify next result, we give the following well-known lemma.

**Lemma 1.** *Let  $A$  be a universally complete vector lattice with a weak order unit  $e$ . Then there exists a unique multiplication on  $A$  such that  $A$  is an  $f$ -algebra with  $e$  as a unit element.*

DEFINITION 2. A vector lattice  $A$  is said to be *hyper-Archimedean* if all quotient spaces  $E/I$ , where  $I$  is an order ideal in  $A$ , are Archimedean.

Several characterizations of hyper-Archimedean vector lattices are known (see, for example, [10], [11, Theorem 37.6, 61.1 and 61.2]). We collect some of them in the following lemma.

**Lemma 2.** *A vector lattice  $A$  is hyper-Archimedean if and only if any of the following equivalent conditions holds.*

- (i) Every prime ideal in  $A$  is a maximal ideal.
- (ii) Every ideal in  $A$  is uniformly closed.
- (iii) The span of the set of all components of  $u$  is the principal ideal generated by  $u$  for all  $u \in A^+$ .

DEFINITION 3. A lattice-ordered algebra  $A$  is called *Artinian* (respectively, *super Artinian*) if  $A$  satisfies the descending chain condition on order ideals (respectively, on bands).

DEFINITION 4. A lattice-ordered algebra  $A$  is called *Noetherian* (respectively, *super Noetherian*) if  $A$  satisfies the ascending chain condition on order ideals (respectively, on bands).

DEFINITION 5. Let  $A$  be a vector lattice. An element  $x$  of  $A$  is said to be *super atomic* if the order ideal  $A_x$  generated by  $x$  is of finite dimension.

We now give the following result which shows the relation between the dimensions of a vector lattice and its universal completion.

**Proposition 3.** *Let  $A$  be a vector lattice and  $A^u$  be its universal completion. Then  $A$  is finite dimension if and only if  $A^u$  is finite dimension.*

$\triangleleft$  Let  $A$  be a finite dimension vector lattice. Since  $A$  is finite dimension and the elements of any finite orthogonal system of  $A$  are linearly independent, it follows that  $A$  has  $\{e_1, e_2, \dots, e_n\}$  as a maximal orthogonal system of length  $n$ . Let  $y \in \{e_i\}^{dd}$  for a fixed index  $i$  with  $1 \leq i \leq n$ . Then  $y = y^+ - y^-$  with  $y^+, y^- \in \{e_i\}^{dd}$ . If  $y^+ \neq 0$  and  $y^- \neq 0$ , it is easy to see that the system  $\{e_1, e_2, \dots, e_{i-1}, y^+, y^-, e_{i+1}, \dots, e_n\}$  is an orthogonal system of length  $n + 1$ , which

is a contradiction. Hence,  $y^+ = 0$  or  $y^- = 0$ . Then, the band  $\{e_i\}^{dd}$  is totally ordered for all  $1 \leq i \leq n$ . By [3, Proposition 1],  $\{e_i\}^{dd} = \mathbb{R}e_i$  for all  $1 \leq i \leq n$ . Hence, the band generated by  $e_i$  in  $A^u$  will be equal to  $\mathbb{R}e_i$ . Then  $A^u = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \dots \oplus \mathbb{R}e_n$  for some  $n \in \mathbb{N}$ . Hence  $A^u$  is finite dimension. Conversely, if  $A^u$  is finite dimension, then so is clearly  $A$ .  $\triangleright$

REMARK 1. If a vector lattice  $A$  is finite dimension then  $A = A^u$ .

We now have all ingredients to give the first main result of this section.

**Theorem 3.** *Let  $A$  be a vector lattice and  $A^u$  be its universal completion. Then the followings are equivalent:*

- (1)  $A$  has the projection property and its depth is finite.
- (2)  $A = I_1 \oplus I_2 \oplus \dots \oplus I_n$  for some simple order ideals  $I_1, I_2, \dots, I_n$  in  $A$ , and  $n \in \mathbb{N}$ .
- (3)  $A^u$  is super Dedekind complete and Artinian.
- (4)  $A^u$  is super Dedekind complete and super Artinian.
- (5)  $A^u$  is super Dedekind complete and Noetherian.
- (6)  $A^u$  is super Dedekind complete and super Noetherian.
- (7)  $A^u$  has at least one super atomic weak order unit.
- (8)  $A$  is finite dimension.

$\triangleleft$  (1)  $\Rightarrow$  (2) This follows from Propositions 1 and 2.

(2)  $\Rightarrow$  (3) Since  $A = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \dots \oplus \mathbb{R}x_n$  for some  $n \in \mathbb{N}$  and some elements  $x_1, x_2, \dots, x_n \in A$ , it follows that  $A^u = A$ . Hence, the set of all order ideals of  $A$  is finite and then  $A$  is Artinian.

(3)  $\Rightarrow$  (4) This path is trivial.

(4)  $\Rightarrow$  (6) Let  $(B_n)_{n \in \mathbb{N}}$  be an increasing sequence of bands in  $A^u$ . Then  $(B_n^d)_{n \in \mathbb{N}}$  is a decreasing sequence of bands in  $A^u$ . By the fact that  $A$  is super Artinian, it follows that there exists  $n_0 \in \mathbb{N}$  such that  $B_n^d = B_{n_0}^d$  for all  $n \geq n_0$ . Consequently,  $B_n = B_{n_0}$  for all  $n \geq n_0$ . Therefore  $A^u$  is super Noetherian.

(6)  $\Rightarrow$  (7) Let  $S = \{e_i : i \in I\}$  be a maximal orthogonal system in  $A^u$ . Then  $e = \sup \{e_i : i \in I\}$  is a weak order unit of  $A^u$ . Since  $A^u$  is super Dedekind complete, it follows that there exists an at most countable subset  $T = \{e_n : n \in \mathbb{N}\}$  of  $S$  such that  $e = \sup \{e_n : n \in \mathbb{N}\}$ . Let  $(B_n)_{n \in \mathbb{N}}$  be an increasing sequence of bands in  $A^u$  where  $B_n = \{\bigvee_{1 \leq i \leq n} e_i\}^{dd}$ . Since  $A^u$  is super Noetherian, it follows that there exists  $n_0 \in \mathbb{N}$  such that  $B_n = B_{n_0}$  for all  $n \geq n_0$ . Consequently,  $\{\bigvee_{1 \leq i \leq n_0} e_i\}^{dd} = \{\bigvee_{1 \leq i \leq n} e_i\}^{dd}$  for all  $n \geq n_0$ . Therefore,  $e_n = 0$  for all  $n > n_0$  and then  $S$  is a finite set. Let  $S' = \{f_{j_n}, j_n \in J_n\}$  be an orthogonal system of  $A^u$  such that  $e_n = \sup \{f_{j_n}, j_n \in J_n\}$  for all  $1 \leq n \leq n_0$ . Since the set  $S' = \{f_{j_n}, j_n \in J_n, 1 \leq n \leq n_0\}$  is a maximal orthogonal system of  $A^u$ , it follows that  $S'$  is a finite set. Using the same argument with each  $f_{j_n}$ , we deduce that there exist  $m_0 \in \mathbb{N}$  and a maximal orthogonal system  $K = \{k_n, 1 \leq n \leq m_0\}$  of  $A^u$  with  $e = \sup \{k_n, 1 \leq n \leq m_0\}$  such that  $K$  will be the finest orthogonal system meaning that we cannot use the decomposition process another time. Consequently,  $A_e^u = A_{k_1}^u \oplus A_{k_2}^u \oplus \dots \oplus A_{k_{m_0}}^u$ . Let  $1 \leq n \leq n_0$  and let  $0 \leq x \leq k_n$ . Let  $H_{k_n} = \{y \in A_{k_n}^u, y = \sum_{1 \leq i \leq m} \alpha_i h_i \text{ where } \alpha_i \in \mathbb{R} \text{ and } h_i \text{ is a component of } k_n\}$ . It is not hard to prove that  $H_e$  is a hyper-Archimedean vector sublattice of  $A^u$ . Since  $K$  is a finest orthogonal system, it follows that the set of all components of  $k_n$  is  $\{0, k_n\}$ .

By Freudenthal Spectral Theorem, there exists  $x_m = \sum_{1 \leq i \leq m} \alpha_i h_i = \alpha_m k_n$  with  $x_n \rightarrow x$  ( $r.u.$ ). Therefore, there exists  $\alpha_n \in \mathbb{R}$  such that  $x = \alpha_n k_n$ . Hence,  $H_{k_n} = A_{k_n}^u$  for all  $1 \leq n \leq n_0$ .

Since any relatively uniformly complete hyper-Archimedean vector lattice is of finite dimension (see [11, Theorems 37.6, 61.4], [12, Theorem 3]), and  $A_{k_n}^u$  is relatively uniformly complete, it follows that  $A_e^u$  is of finite dimension. Therefore,  $e$  is super atomic.

(7)  $\Rightarrow$  (8) Let  $e$  be a super atomic weak order unit of  $A^u$ . Then  $A_e^u$  is of finite dimension. Since  $x \wedge ne \rightarrow x$  ( $r.u$ ) for all  $0 \leq x \in A^u$ , it follows that  $A^u$  is of finite dimension.

(8)  $\Leftrightarrow$  (1) This equivalence is trivial.

(1)  $\Rightarrow$  (5) Since  $A^u$  is of finite dimension, the set of all order ideals of  $A^u$  is finite and then  $A^u$  satisfies the ascending chain condition on ideals and so we are done.  $\triangleright$

Let  $A$  be a vector lattice and  $E$  be a vector sublattice of  $A$ . A positive linear operator  $\pi : E \rightarrow A$  is said to be a *positive orthomorphism* if  $\pi(x) \wedge y = 0$  whenever  $x \wedge y = 0$  for each  $x, y \in E$ . An *orthomorphism* is the difference of two positive orthomorphisms.  $\text{Orth}(E, A)$  will denote the vector lattice of all orthomorphisms from  $E$  to  $A$ .  $Z(E, A)$  will denote the sublattice of  $\text{Orth}(E, A)$  consisting of those  $\pi$  for which there is a non-negative real  $\lambda$  with

$$-\lambda x \leq \pi(x) \leq \lambda x$$

for all  $x \in E^+$ . Let  $A$  be a lattice-ordered algebra.  $M(A)$  denotes the set of all maximal two-sided algebra ideals. We consider a subset  $m(A)$  of  $M(A)$  consisting of relatively uniformly closed ideals.

All prerequisites are made for the second main result of this section.

**Theorem 4.** *Let  $A$  be a vector lattice and let  $A^u$  be its universal completion unital  $f$ -algebra. Then the following conditions are equivalent:*

- (1)  $A$  is finite dimension.
- (2) Every maximal modular algebra ideal in  $A^u$  is relatively uniformly closed.
- (3)  $\text{Orth}(A, A^u) = Z(A, A^u)$ .

$\triangleleft$  (1)  $\Rightarrow$  (2) Since  $A$  is of finite dimension, then  $A^u$  is of finite dimension so that  $A^u$  becomes a Banach lattice. It is well-known that any maximal modular algebra ideal of a commutative Banach algebra is closed. So we are done.

(2)  $\Rightarrow$  (3) Since any maximal modular algebra ideal of  $A^u$  is relatively uniformly closed and by using the main result of [13], we deduce that  $\text{Orth}(A^u) = Z(A^u)$ . Moreover, it is not hard to prove that  $\text{Orth}(A^u) = \text{Orth}(A, A^u)$  and  $Z(A, A^u) = Z(A^u)$ .

(3)  $\Rightarrow$  (1) Since  $\text{Orth}(A^u) = \text{Orth}(A, A^u)$ ,  $Z(A, A^u) = Z(A^u)$  and  $\text{Orth}(A^u) = Z(A, A^u)$ , it follows that  $\text{Orth}(A^u) = Z(A^u) = A^u$ . Consequently,  $A^u$  becomes a Banach lattice and it is well-known that any Banach universally complete vector lattice is of finite dimension. Hence,  $A$  is of finite dimension and we are done.  $\triangleright$

It is well-known that any universally complete vector lattice  $A$  is of the form  $C^\infty(X)$  for some Hausdorff extremally disconnected compact topological space  $X$  (i. e. the closure of every open set of  $X$  is also open). The symbol  $C^\infty(X)$  denotes the collection of all continuous functions  $f : X \rightarrow [-\infty, +\infty]$  for which the open set  $\text{dom } f = \{x \in X : -\infty < f(x) < +\infty\}$  is dense in  $X$ . It is well-known that  $C^\infty(X)$  can be equipped with a unital  $f$ -algebra multiplication. Moreover,  $C^\infty(X)$  is a Dedekind complete  $f$ -algebra with  $e := \chi_X$  as a unit element. The orthomorphisms in  $C^\infty(X)$  are the pointwise multiplications, so  $\text{Orth}(C^\infty(X)) = C^\infty(X)$ .

In order to reach our aim, we need the following;

**Theorem 5** [14, 15]. *Given an extremally disconnected compact space  $X$ , the following properties of a point  $x \in X$  are pairwise equivalent:*

- (1) The intersection of every sequence of neighborhoods of  $x$  is a neighborhood of  $x$ .
- (2)  $x \in \text{dom } f$  for all  $f \in C^\infty(X)$ .
- (3) If  $f \in C^\infty(X)$  and  $f(x) = 0$ , then  $f \equiv 0$  in some neighborhood of  $x$ .

DEFINITION 6. The point  $x$  is called  $\sigma$ -isolated (or a  $P$ -point, or bounded) whenever  $x$  enjoys any of the properties in Theorem 5.

**Theorem 6** [14, 15]. *The maximal algebra ideals of  $C^\infty(X)$  for  $x \in X$  are of the form  $(C^\infty(X))_x := \{f \in C^\infty(X) : f \equiv 0 \text{ in some neighborhood of } x\}$ .*

We now have gathered all ingredient for the third result of this section.

**Theorem 7.** *Let  $A$  be a vector lattice and let  $A^u (= C^\infty(X))$  be universal completion unital  $f$ -algebra of  $A$ . Then the following conditions are equivalent:*

- (1)  $A$  is of finite dimension.
- (2) Each element of  $X$  is  $\sigma$ -isolated.

$\triangleleft$  (1)  $\Rightarrow$  (2) Since  $A$  is of finite dimension, then  $A^u$  is of finite dimension; therefore  $A^u$  becomes a Banach unital  $f$ -algebra and it is well-known that any maximal algebra ideal of a commutative Banach algebra is closed. By Theorem 6, any maximal algebra ideal is of the form  $(C_\infty(X))_x$  for some  $x \in X$  and since  $(C_\infty(X))_x$   $(C_\infty(X))_x$  is relatively uniformly closed, then  $x$  is  $\sigma$ -isolated (see [14, 15]).

(1)  $\Rightarrow$  (2) Since any point of  $X$  is  $\sigma$ -isolated, it follows that any maximal algebra ideal of  $A^u$  is relatively uniformly closed (see [14, 15]). By the main result of [13], we deduce that  $\text{Orth}(A^u) = Z(A^u)$ . Since  $\text{Orth}(A^u) = Z(A^u) = A^u$ ,  $A^u$  becomes a Banach lattice and it is well-known that any Banach universally complete vector lattice is of finite dimension. Hence,  $A$  is of finite dimension and we are done.  $\triangleright$

Next, we will give a new class of non-finite dimension zero product algebras for which the characterization of Theorem 2 holds.

**Theorem 8.** *Let  $A$  be an Archimedean unital  $f$ -algebra. Then  $A$  is zero product determined if and only if  $A$  is hyper-Archimedean.*

$\triangleleft$  For the proof of “only if” part, we will use the same argument as in [16, Theorem 8]. Assume that every bilinear map  $f : A \times A \rightarrow B$ , where  $B$  is an arbitrary vector space over  $\mathbb{K}$ , with the property that for all  $x, y \in A$ ,  $f(x, y) = 0$  whenever  $xy = 0$ , is of the form  $f(x, y) = \Phi(xy)$  for some linear map  $\Phi : A \rightarrow B$ . Suppose, per contra, that  $A$  is not hyper-Archimedean. It follows that there exists a prime ideal  $I$  which is not maximal. Hence the quotient  $A/I$  is linearly ordered space that is not isomorphic to  $\mathbb{R}$  (see [11, Theorem 27.3 and 33.2]). Let  $\bar{x}, \bar{y}$  in  $A/I$  that are linearly independent. Hence  $\bar{x} + \bar{y}, \bar{x}$  are linearly independent. By using Zorn’s Lemma, it is not hard to prove that  $\bar{x}, \bar{y}$  are contained in a Hamel basis  $H_1$  and  $\bar{x} + \bar{y}, \bar{x}$  are contained in a Hamel basis  $H_2$  such that  $H_1 \neq H_2$ . Then there exist two linear maps  $f : A/I \rightarrow \mathbb{R}$  such that  $f(\bar{x}) = 1$  and  $f(\bar{y}) = -1$  and  $g : A/I \rightarrow \mathbb{R}$  such that  $g(\bar{x} + \bar{y}) = 1$  and  $g(\bar{y}) = -1$ . Let the bilinear map  $\Psi : A \times A \rightarrow \mathbb{R}$  be defined by  $\Psi(a, b) = f(\bar{a})g(\bar{b})$ , for all  $a, b \in A$ . Let  $a, b \in A$  such that  $ab = 0$ . Since  $I$  is a prime ideal, it follows that  $a \in I$  or  $b \in I$ . Hence  $\bar{a} = \bar{0}$  or  $\bar{b} = \bar{0}$ . Consequently,  $f(\bar{a}) = 0$  or  $g(\bar{b}) = 0$ . Therefore  $\Psi(a, b) = 0$ . Hence,  $\Psi$  is orthosymmetric. Whereas,  $\Psi(x, y) = f(\bar{x})g(\bar{y}) \neq f(\bar{y})g(\bar{x}) = \Psi(y, x)$ . That is  $\Psi$  is not symmetric. Hence  $\Psi$  is not of the form  $\Phi(xy)$  for some linear map  $\Phi : A \rightarrow \mathbb{R}$ , which is a contradiction.

Then “if” part remains. We will use the same argument as in [17, Theorem 1]. Let  $\Psi : A \times A \rightarrow B$ , where  $B$  is an arbitrary vector space over  $\mathbb{K}$ , with the property that for all  $x, y \in A$ ,  $\Psi(x, y) = 0$  whenever  $xy = 0$ .

Let  $x, y \in A$ . It follows that  $x = \sum_{i=1}^n \alpha_i e_i$  and  $y = \sum_{j=1}^m \beta_j f_j$ , where  $e_i$  and  $f_j$  are components of  $e = |x| + |y|$ . Then

$$\Psi(x, y) = \sum_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}} \alpha_i \beta_j \Psi(e_i, f_j). \quad (1)$$

Let  $e_i^d$  be the disjoint complement of  $e_i$ . Hence  $e = e_i + e_i^d$  where  $e_i \wedge e_i^d = 0$ . Then

$$f_j = f_j \wedge e = f_j \wedge (e_i + e_i^d) = (f_j \wedge e_i) + (f_j \wedge e_i^d).$$

Since  $(f_j \wedge e_i^d) \wedge e_i = 0$ , then  $(f_j \wedge e_i^d) e_i = 0$ . Consequently,

$$\Psi(e_i, f_j) = \Psi(e_i, (f_j \wedge e_i) + (f_j \wedge e_i^d)) = \Psi(e_i, f_j \wedge e_i).$$

Moreover  $e_i = e_i \wedge e = e_i \wedge (f_j + f_j^d) = (f_j \wedge e_i) + (e_i \wedge f_j^d)$ . Hence,

$$\Psi(e_i, f_j) = \Psi(e_i, f_j \wedge e_i) = \Psi((f_j \wedge e_i) + (e_i \wedge f_j^d), f_j \wedge e_i).$$

Since  $(e_i \wedge f_j^d) \wedge (f_j \wedge e_i) = 0$ , it follows that  $(e_i \wedge f_j^d)(f_j \wedge e_i) = 0$  and then

$$\Psi(e_i, f_j) = \Psi(f_j \wedge e_i, f_j \wedge e_i) = \Psi(f_j, e_i)$$

By using the same argument, we prove that

$$\Psi(e_i, f_j) = \Psi(e_i, f_j \wedge e_i) = \Psi(f_j \wedge e_i, f_j \wedge e_i) = \Psi(f_j, e_i).$$

Therefore, in view of equality (1), we have

$$\Psi(x, y) = \Psi(y, x). \tag{2}$$

Let  $u$  be the unit of  $A$  and let  $z \in A$ . Then the following bilinear map  $\Psi_z : A \times A \rightarrow B$  defined by  $\Psi_z(x, y) = \Psi(xz, y)$ , for all  $x, y \in A$ , satisfies the property that  $\Psi(x, y) = 0$  whenever  $xy = 0$ . Therefore, by using the argument as for  $\Psi$ , we deduce that  $\Psi_z(x, y) = \Psi_z(y, x) = \Psi_x(z, y) = \Psi_x(y, z)$ , for all  $x, y \in A$ . In particular if  $z = u$ , it follows that  $\Psi(x, y) = \Psi_x(y, e) = \Psi(xy, e)$ . Consequently,  $\Psi$  is not of the form  $\Phi(xy)$ , where  $\Phi : A \rightarrow B$  is defined by  $\Phi(x) = \Psi(x, e)$ , for all  $x \in A$  and we are done.  $\triangleright$

It should be noted that a relatively uniformly complete unital hyper-Archimedean  $f$ -algebra is of finite dimension (see [11, Theorems 37.6, 61.4], [12, Theorem 3]). Consequently, we have the following characterization.

**Theorem 9.** *Let  $A$  be a relatively uniformly unital  $f$ -algebra. Then the following properties are equivalent.*

- (1)  $A$  is zero product determined.
- (2)  $A$  is hyper-Archimedean.
- (3)  $A$  is of finite dimension.

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## ХАРАКТЕРИЗАЦИЯ КОНЕЧНОМЕРНЫХ АРХИМЕДОВЫХ ВЕКТОРНЫХ РЕШЕТОК

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**Аннотация.** Статья посвящена условиям конечномерности архимедовых векторных решеток. Найдены три новые характеристики таких решеток. Первая описывает конечномерность векторной решетки  $A$  на языке ее универсального пополнения  $A^u$ . Вторая утверждает, что векторная решетка конечномерна в том и только в том случае, когда выполнено одно из следующих двух условия: (а) всякий максимальный модулярный алгебраический идеал в  $A^u$  равномерно полон; (б)  $\text{Orth}(A, A^u) = Z(A, A^u)$ , где  $\text{Orth}(A, A^u)$  векторная решетка всех ортоморфизмов из  $A$  в  $A^u$ , а  $Z(A, A^u)$  — подрешетка, состоящая из ортоморфизмов  $\pi$ , удовлетворяющих условию  $|\pi(x)| \leq \lambda|x|$  ( $x \in A$ ) при некотором положительном  $\lambda \in \mathbb{R}$ . Хорошо известно, что всякая универсально полная векторная решетка представляется в виде  $C^\infty(X)$  для некоторого экстремально несвязного компакта  $X$ . Точку  $x \in X$  называют  $\sigma$ -изолированной, если пересечение любой последовательности окрестностей точки  $x$  является окрестностью точки  $x$ . Третья характеристика состоит в том, что векторная решетка  $A$  с универсальным расширением  $A^u = C^\infty(X)$  конечномерна тогда и только тогда, когда каждая точка в  $X$   $\sigma$ -изолирована. В качестве приложения получен положительный ответ на вопрос Брезара о существовании новых примеров алгебр, определяемых нулевыми произведениями.

**Ключевые слова:** векторная решетка, *f*-алгебра, гипер-архимедовость, универсальная полнота.