Dedicated to E. I. Gordon on the occasion of his 70th birthday

Abstract. Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal finite trace $\tau$, and let $S(\mathcal{M}, \tau)$ be an $\tau$-algebra of all $\tau$-measurable operators affiliated with $\mathcal{M}$. For $x \in S(\mathcal{M}, \tau)$ the generalized singular value function $\mu(x) : t \mapsto \mu(t; x)$, $t > 0$, is defined by the equality $\mu(t; x) = \inf\{\|xp\|_\mathcal{M} : p^2 = p^* = p \in \mathcal{M}, \tau(1 - p) \leq t\}$. Let $\psi$ be an increasing concave continuous function on $[0, \infty)$ with $\psi(0) = 0$, $\psi(\infty) = \infty$, and let $\Lambda_\psi(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \|x\|_\psi = \int_0^\infty \mu(t; x)d\psi(t) < \infty\}$ be the non-commutative Lorentz space. A surjective (not necessarily linear) mapping $V : \Lambda_\psi(\mathcal{M}, \tau) \to \Lambda_\psi(\mathcal{M}, \tau)$ is called a surjective 2-local isometry, if for any $x, y \in \Lambda_\psi(\mathcal{M}, \tau)$ there exists a surjective linear isometry $V_{x,y}$ on $\Lambda_\psi(\mathcal{M}, \tau)$ such that $V(x) = V_{x,y}(x)$ and $V(y) = V_{x,y}(y)$. It is proved that in the case when $\mathcal{M}$ is a factor, every surjective 2-local isometry $V : \Lambda_\psi(\mathcal{M}, \tau) \to \Lambda_\psi(\mathcal{M}, \tau)$ is a linear isometry.

Key words: measurable operator, Lorentz space, isometry.

Mathematical Subject Classification (2010): 46L52, 46B04.

equipped with a faithful normal semifinite trace \( \tau \) (see, for example, [4, Ch. 2, § 2.5]). It is natural to expect that for these non-commutative symmetric spaces with the Fatou property, every surjective 2-local isometry \( V : \mathcal{E}(\mathcal{M}, \tau) \rightarrow \mathcal{E}(\mathcal{M}, \tau) \) is a linear map. Unfortunately, the method of proof of a similar statement for Banach ideals \((C_E, \| \cdot \|_{C_E})\) cannot be applied here, since there is no description of surjective linear isometries \( V : \mathcal{E}(\mathcal{M}, \tau) \rightarrow \mathcal{E}(\mathcal{M}, \tau) \). At the same time, in the case of non-commutative Lorentz and Marcinkiewicz spaces, such a description of surjective linear isometries was obtained in the paper [5]. Using this description, we obtain the following description of surjective 2-local isometries of non-commutative Lorentz spaces.

**Theorem 1.** Let \( \mathcal{M} \) be an arbitrary factor with a faithful normal finite trace \( \tau \), and let \((\Lambda_\psi(\mathcal{M}, \tau), \| \cdot \|_\psi)\) be a non-commutative Lorentz space. Then every surjective 2-local isometry \( V : \Lambda_\psi(\mathcal{M}, \tau) \rightarrow \Lambda_\psi(\mathcal{M}, \tau) \) is a linear isometry.

### 2. Preliminaries

Let \( \mathcal{H} \) be an infinite-dimensional complex Hilbert space, let \( \mathcal{B}(\mathcal{H}) \) be the \( C^* \)-algebra of all bounded linear operators in \( \mathcal{H} \), and let \( \mathbf{1} \) be the unit in \( \mathcal{B}(\mathcal{H}) \). Let \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) be a von Neumann algebra on Hilbert space \( \mathcal{H} \) equipped with a faithful normal semifinite trace \( \tau \) (see, for example, [6]). A linear operator \( x : \mathcal{D}(x) \rightarrow \mathcal{H} \), where the domain \( \mathcal{D}(x) \) of \( x \) is a linear subspace of \( \mathcal{H} \), is said to be affiliated with \( \mathcal{M} \) if \( yx \subseteq xy \) for all \( y \in \mathcal{M}' \), where \( \mathcal{M}' \) is the commutant of \( \mathcal{M} \). A linear operator \( x : \mathcal{D}(x) \rightarrow \mathcal{H} \) is termed measurable with respect to \( \mathcal{M} \) if \( x \) is closed, densely defined, affiliated with \( \mathcal{M} \) and there exists a sequence \( \{p_n\}_{n=1}^\infty \) in the lattice \( \mathcal{P}(\mathcal{M}) \) of all projections of \( \mathcal{M} \), such that \( p_n \uparrow \mathbf{1}, p_n(\mathcal{H}) \subseteq \mathcal{D}(x) \) and \( \mathbf{1} - p_n \) is a finite projection (with respect to \( \mathcal{M} \)) for all \( n \). The collection \( S(\mathcal{M}) \) of all measurable operators with respect to \( \mathcal{M} \) is a unital *-algebra with respect to strong sums and products.

Let \( x \) be a self-adjoint operator affiliated with \( \mathcal{M} \) and let \( \{e^x\} \) be a spectral measure of \( x \). It is well known that if \( x \) is a closed operator affiliated with \( \mathcal{M} \) with the polar decomposition \( x = u|x| \), then \( u \in \mathcal{M} \) and \( e \in \mathcal{M} \) for all projections \( e \in \{e^x\} \). Moreover, \( x \in S(\mathcal{M}) \) if and only if \( x \) is closed, densely defined, affiliated with \( \mathcal{M} \) and \( e^x(\lambda, \infty) \) is a finite projection for some \( \lambda > 0 \).

An operator \( x \in S(\mathcal{M}) \) is called \( \tau \)-measurable if there exists a sequence \( \{p_n\}_{n=1}^\infty \) in \( \mathcal{P}(\mathcal{M}) \) such that \( p_n \uparrow \mathbf{1}, p_n(\mathcal{H}) \subseteq \mathcal{D}(x) \) and \( \tau(1 - p_n) < \infty \) for all \( n \). The collection \( S(\mathcal{M}, \tau) \) of all \( \tau \)-measurable operators is a unital *-subalgebra of \( S(\mathcal{M}) \). It is well known that a linear operator \( x \) belongs to \( S(\mathcal{M}, \tau) \) if and only if \( x \in S(\mathcal{M}) \) and there exists \( \lambda = \lambda(x) > 0 \) such that \( \tau(e^x(\lambda, \infty)) < \infty \).

The generalized singular value function \( \mu(x) : t \rightarrow \mu(t; x), t > 0 \), of the operator \( x \in S(\mathcal{M}, \tau) \) is defined by setting \([7]\)

\[
\mu(t; x) = \inf \left\{ \|xp\| : p \in \mathcal{P}(\mathcal{M}), \tau(1 - p) \leq t \right\} = \inf \left\{ s > 0 : \tau(e^x(s, \infty)) \leq t \right\}.
\]

A non-zero linear subspace \( \mathcal{E}(\mathcal{M}, \tau) \subset S(\mathcal{M}, \tau) \) with the Banach norm \( \| \cdot \|_{\mathcal{E}(\mathcal{M}, \tau)} \) is called a symmetric space if the conditions

\[
x \in \mathcal{E}(\mathcal{M}, \tau), \quad y \in S(\mathcal{M}, \tau), \quad \mu_t(y) \leq \mu_t(x) \quad \text{for all} \quad t > 0,
\]

imply that \( y \in \mathcal{E}(\mathcal{M}, \tau) \) and \( \|y\|_{\mathcal{E}(\mathcal{M}, \tau)} \leq \|x\|_{\mathcal{E}(\mathcal{M}, \tau)} \).

It is known that in the case \( \tau(1) < \infty \) it is true

\[
S(\mathcal{M}) = S(\mathcal{M}, \tau) \quad \text{and} \quad \mathcal{M} \subseteq \mathcal{E}(\mathcal{M}, \tau) \subseteq L_2(\mathcal{M}, \rho).
\]
for each symmetric space $E(\mathcal{M},\tau)$, where

$$L_1(\mathcal{M},\tau) = \left\{ x \in S(\mathcal{M},\tau) : \|x\|_1 = \int_0^\infty \mu_t(x) dt < \infty \right\}. $$

In addition,

$$\mathcal{M} \cdot E(\mathcal{M},\tau) \cdot \mathcal{M} \subseteq E(\mathcal{M},\tau),$$

and

$$\|axb\|_{E(\mathcal{M},\tau)} \leq \|a\|_{\mathcal{M}} \cdot \|b\|_{\mathcal{M}} \cdot \|x\|_{E(\mathcal{M},\tau)}$$

for all $a,b \in \mathcal{M}$, $x \in E(\mathcal{M},\tau)$.

Let $\psi$ be an increasing concave continuous function on $[0,\infty)$ with $\psi(0) = 0$, $\psi(\infty) = \lim_{t \to \infty} \psi(t) = \infty$, and let

$$\Lambda_\psi(\mathcal{M},\tau) = \left\{ x \in S(\mathcal{M},\tau) : \|x\|_\psi = \int_0^\infty \mu(t;x) d\psi(t) < \infty \right\}$$

be the non-commutative Lorentz space. It is known that $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$ is a symmetric space [8], and the norm $\|\cdot\|_\psi$ has the Fatou property, that is, the conditions $0 \leq x_k \in \Lambda_\psi(\mathcal{M},\tau)$ for all $k$, and $\sup_{k \geq 1} \|x_k\|_\psi < \infty$, imply that there exists $0 \leq x \in \Lambda_\psi(\mathcal{M},\tau)$ such that $x_k \uparrow x$ and $\|x\|_\psi = \sup_{k \geq 1} \|x_k\|_\psi$.

Denote by $M_\psi(\mathcal{M},\tau)$ the set of all $x \in S(\mathcal{M},\tau)$ for which

$$\|x\|_{M_\psi} = \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu(s;x) ds$$

is finite. The set $M_\psi(\mathcal{M},\tau)$ with the norm $\|\cdot\|_{M_\psi}$ is a symmetric space which is called a Marcinkiewicz space.

Denote by $M_\psi^0(\mathcal{M},\tau)$ the closure of $M$ in $M_\psi(\mathcal{M},\tau)$. It is known [9] that the conjugate space of $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$ is identified with $(M_\psi(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$, and the conjugate space of $(M_\psi^0(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$, under the condition $\lim_{t \to 0+} \frac{t}{\psi(t)} = 0$, is identified with $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$. The duality in these pairs of spaces is realized via the bilinear form $(x,y) = \tau(xy)$. It should be pointed out that the spaces $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$, $(M_\psi(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$ and $(M_\psi^0(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$ are symmetric spaces [4, Ch. 2, §2.6], [8].

3. Isometries of Non-Commutative Lorentz Spaces

Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra on Hilbert space $\mathcal{H}$. A linear bijective mapping $\Phi : \mathcal{M} \to \mathcal{M}$ is called a Jordan isomorphism if $\Phi(x^2) = (\Phi(x))^2$ and $\Phi(x^*) = (\Phi(x))^*$ for all $x \in \mathcal{M}$.

If $\Phi : \mathcal{M} \to \mathcal{M}$ is a Jordan isomorphism, then there exists a central projection $z \in \mathcal{M}$ such that $\Phi_+(x) = \Phi(z \cdot x \cdot z)$, $x \in \mathcal{M}$, is an $*$-homomorphism, and $\Phi_-(x) = \Phi(x) \cdot (1-z)$, $x \in \mathcal{M}$, is an $*$-antihomomorphism (see, for example, [10, Ch. 3, §3.2.1]). Consequently, if $\mathcal{M}$ is a factor then a Jordan isomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ is an $*$-homomorphism or $*$-antihomomorphism.

If $\tau$ is a faithful normal finite trace on von Neumann algebra $\mathcal{M}$ then a Jordan isomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ is continuous with respect to measure topology $t_\tau$ generated by trace $\tau$ (see, for
example, [11, Ch. 5, §3, Proposition 1]). Therefore, \( \Phi \) extends to a \( t_\tau \)-continuous Jordan isomorphism \( \hat{\Phi}: S(M, \tau) \to S(M, \tau) \). In addition, if \( \tau(\Phi(x)) = \tau(x) \) for all \( x \in M \) then \( \mu(t; \Phi(x)) = \mu(t; x) \) for all \( t \in S(M, \tau) \), in particular, \( \Phi(E(M, \tau)) = E(M, \tau) \) and \( \|\Phi(x)\|_{E(M, \tau)} = \|x\|_{E(M, \tau)} \) for all \( x \in E(M, \tau) \), that is, \( \hat{\Phi} : E(M, \tau) \to E(M, \tau) \) is a surjective linear isometry for any symmetric space \( (E(M, \tau), \| \cdot \|_{E(M, \tau)} ) \).

Thus, it is true the following

**Proposition 1.** Let \( M \) be an arbitrary von Neumann algebra with a faithful normal finite trace \( \tau \), and let \( \Phi : M \to M \) be a Jordan isomorphism such that \( \tau(\Phi(x)) = \tau(x) \) for all \( x \in M \). Then for every symmetric space \( (E(M, \tau), \| \cdot \|_{E(M, \tau)} ) \) the mapping \( \hat{\Phi} : E(M, \tau) \to E(M, \tau) \) given by the equality \( \hat{\Phi}(x) = u \cdot \Phi(x) \cdot v \), \( x \in E(M, \tau) \), \( u, v \) are unitary operators in \( M \), is a surjective linear isometry.

We need the following description of surjective linear isometries of the spaces \( (\Lambda_0(M, \tau), \| \cdot \|_\tau) \) and \( (M^0_\psi(M, \tau), \| \cdot \|_{M_\psi}) \) [5, Theorems 5.1, 6.1].

**Theorem 2.** Let \( M \) be an arbitrary von Neumann algebra with a faithful normal finite trace \( \tau \), and let \( V : \Lambda_0(M, \tau) \to \Lambda_0(M, \tau) \) (respectively, \( V : M^0_\psi(M, \tau) \to M^0_\psi(M, \tau) \)) be a surjective linear isometry. Then there exist uniquely an unitary operator \( u \in M \) and a Jordan isomorphism \( \Phi : M \to M \) such that \( V(x) = u \cdot \Phi(x) \) and \( \tau(\Phi(x)) = \tau(x) \) for all \( x \in M \).

## 4. Local Isometries of Non-Commutative Lorentz Spaces

Let \( (X, \| \cdot \|_\chi) \) be an arbitrary Banach space over the field \( \mathbb{K} \) of complex or real numbers. A surjective (not necessarily linear) mapping \( T : X \to X \) is called a surjective 2-local isometry \([2] \), if for any \( x, y \in X \) there exists a surjective linear isometry \( V_{x,y} : X \to X \) such that \( T(x) = V_{x,y}(x) \) and \( T(y) = V_{x,y}(y) \). It is clear that every surjective linear isometry on \( X \) is a surjective 2-local isometry on \( X \). In addition,

\[
T(\lambda x) = V_{x,\lambda x}(\lambda x) = \lambda V_{x,x}(\lambda x) = \lambda T(x)
\]

for any \( x \in X \) and \( \lambda \in \mathbb{K} \).

Consequently, in order to establish linearity of a 2-local isometry \( T \), it is sufficient to show that \( T(x + y) = T(x) + T(y) \) for all \( x, y \in X \).

Since

\[
\|T(x) - T(y)\|_\chi = \|V_{x,y}(x) - V_{x,y}(y)\|_\chi = \|x - y\|_\chi \quad \text{for all} \quad x, y \in X,
\]

it follows that \( T \) is continuous map on \( (X, \| \cdot \|_\chi) \). In addition, in the case a real Banach space \( X (\mathbb{K} = \mathbb{R}) \), every surjective 2-local isometry on \( X \) is a linear map (see Mazur–Ulam Theorem \([12, \text{Ch.} 1, \S 1.3, \text{Theorem} 1.3.5]) \). In the case a complex Banach space \( X (\mathbb{K} = \mathbb{C}) \), this fact is not true.

Using the description of all surjective linear isometries on a separable Banach symmetric ideal \( C_E \) \([3] \) (respectively, on a Banach symmetric ideal \( C_E \) with Fatou property \([1]\) ), \( C_E \neq C_{l_2} \), in the papers \([1, 2]\) it is proved that every surjective 2-local isometry \( T : C_E \to C_E \) is a linear isometry.

The following Theorem is a version of the above results for the spaces \( \Lambda_0(M, \tau) \) and \( M^0_\psi(M, \tau) \).

**Theorem 3.** Let \( M \) be an arbitrary factor with a faithful normal finite trace \( \tau \), and let \( T : \Lambda_0(M, \tau) \to \Lambda_0(M, \tau) \) (respectively, \( T : M^0_\psi(M, \tau) \to M^0_\psi(M, \tau) \)) be a surjective 2-local isometry. Then \( T \) is a linear isometry.
Fix \( x, y \in \mathcal{M} \) and let \( V_{x,y} : \Lambda_\psi(\mathcal{M}, \tau) \rightarrow \Lambda_\psi(\mathcal{M}, \tau) \) be a surjective isometry such that \( T(x) = V_{x,y}(x) \) and \( T(y) = V_{x,y}(y) \). By Theorem 2, there exist uniquely an unitary operator \( u \in \mathcal{M} \) and a Jordan isomorphism \( \Phi : \mathcal{M} \rightarrow \mathcal{M} \) such that \( V_{x,y}(a) = u \cdot \Phi(a) \) and \( \tau(\Phi(a)) = \tau(a) \) for all \( a \in \mathcal{M} \). Since \( \mathcal{M} \) is a factor it follows then \( \Phi : \mathcal{M} \rightarrow \mathcal{M} \) is an \(*\)-isomorphism or \( \Phi \) is an \(*\)-anti-isomorphism.

We assume that \( \Phi \) is an \(*\)-isomorphism (in the case when \( \Phi \) is an \(*\)-anti-isomorphism, the proof is similar).

We have

\[
\tau(T(x) \cdot (T(y))^*) = \tau(V_{x,y}(x) \cdot (V_{x,y}(y))^*)
\]

\[
= \tau(u \cdot \Phi(x) \cdot (u \cdot \Phi(y))^*) = \tau(u \cdot \Phi(xy^*) \cdot u^*) = \tau(\Phi(xy^*)) = \tau(xy^*).
\]

Consequently, \( \tau(T(x) \cdot (T(y))^*) = \tau(xy^*) \) for all \( x, y \in \mathcal{M} \).

If \( x, y, z \in \mathcal{M} \), then

\[
\tau(T(x + y) \cdot (T(z))^*) = \tau((x + y)z^*), \quad \tau(T(x) \cdot T(z)^*) = \tau(xz^*),
\]

\[
\tau(T(y) \cdot T(z)^*) = \tau(y \cdot z^*).
\]

Therefore

\[
\tau((T(x + y) - T(x) - T(y)) \cdot (T(z))^*) = 0
\]

for all \( z \in \mathcal{M} \). Taking \( z = x + y, \ z = x \) and \( z = y \), we obtain

\[
\tau((T(x + y) - T(x) - T(y)) \cdot ((T(x + y) - T(x) - T(y))^*) = 0,
\]

that is, \( T(x + y) = T(x) + T(y) \) for all \( x, y \in \mathcal{M} \).

Since the Lorentz space \( \Lambda_\psi(0, \infty) \) of measurable functions on a semi-axis \([0, \infty)\) is separable space [13, Ch. 21, §5], it follows that the non-commutative Lorentz \( (\Lambda_\psi(\mathcal{M}, \tau), \| \cdot \|_\psi) \) has an order continuous norm [14, Proposition 3.6], that is, \( \| x_n \|_\psi \downarrow 0 \) whenever \( x_n \in \Lambda_\psi(\mathcal{M}, \tau) \) and \( x_n \downarrow 0 \). Consequently, the factor \( \mathcal{M} \) is dense in the space \( \Lambda_\psi(\mathcal{M}, \tau) \). Since \( T \) is a continuous mapping on \( \Lambda_\psi(\mathcal{M}, \tau) \) it follows that \( T(x + y) = T(x) + T(y) \) for all \( x, y \in \Lambda_\psi(\mathcal{M}, \tau) \), that is, \( T \) is a surjective linear isometry. For the space \( M_\psi^0(\mathcal{M}, \tau) \), the proof of the linearity of the surjective 2-local isometry \( T : M_\psi^0(\mathcal{M}, \tau) \rightarrow M_\psi^0(\mathcal{M}, \tau) \) repeats the previous proof. △

References


2-ЛОКАЛЬНЫЕ ИЗОМЕТРИИ НЕКОММУТАТИВНЫХ ПРОСТРАНСТВ ЛОРЕНЦА

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Аннотация. Пусть $M$ алгебра фон Неймана с точным нормальным конечным следом $\tau$, и пусть $S(M,\tau)$ инволютивная алгебра всех $\tau$-измеримых операторов, присоединенных к алгебре $M$. Для оператора $x \in S(M,\tau)$ невозрастающая перестановка $\mu(x) : t \to \mu(t;x)$, $t > 0$, определяется с помощью равенства $\mu(t;x) = \inf\{\|xp\|_M : p^2 = p^* = p \in M, \tau(1-p) \leq t\}$. Пусть $\psi$ возрастающая вогнутая непрерывная функция на $[0,\infty)$, для которой $\psi(0) = 0, \psi(\infty) = \infty$. Пусть $\Lambda_{\psi}(M,\tau) = \{x \in S(M,\tau) : \|x\|_\psi = \int_0^\infty \mu(t;x)d\psi(t) < \infty\}$ некоммутативное пространство Лоренца. Сюръективное (не обязательно линейное) отображение $V : \Lambda_{\psi}(M,\tau) \to \Lambda_{\psi}(M,\tau)$ называется сюръективной 2-локальной изометрией, если для любых $x,y \in \Lambda_{\psi}(M,\tau)$ существует такая сюръективная линейная изометрия $V_{x,y} : \Lambda_{\psi}(M,\tau) \to \Lambda_{\psi}(M,\tau)$, что $V(x) = V_{x,y}(x)$ и $V(y) = V_{x,y}(y)$. Доказано, что в случае, когда $M$ есть фактор, каждая сюръективная 2-локальная изометрия $V : \Lambda_{\psi}(M,\tau) \to \Lambda_{\psi}(M,\tau)$ есть линейная изометрия.

Ключевые слова: измеримый оператор, пространство Лоренца, изометрия.

Mathematical Subject Classification (2010): 46L52, 46B04.