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BOUNDED COMPOSITION OPERATORS ON WEIGHTED FUNCTION SPACES IN THE UNIT DISK

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Dedicated to Professor Yu. F. Korobeinik
on the occasion of his 90th birthday

Abstract. We introduce a general class of weighted spaces $\mathscr{H}(\beta)$ of holomorphic functions in the unit disk $\mathbb{D}$, which contains several classical spaces, such as Hardy space, Bergman space, Dirichlet space. We characterize boundedness of composition operators C_φ induced by affine and monomial symbols φ on these spaces $\mathscr{H}(\beta)$. We also establish a sufficient condition under which the operator C_φ induced by the symbol φ with relatively compact image $\varphi(\mathbb{D})$ in $\mathbb{D}$ is bounded on $\mathscr{H}(\beta)$. Note that in the setting of $\mathscr{H}(\beta)$, the characterizations of boundedness of composition operators C_φ depend closely not only on functional properties of the symbols φ but also on the behavior of the weight sequence β.

Key words: composition operator, weighted space, weight sequence, holomorphic function, unit disk.

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1. Introduction

Let $\mathscr{H}$ be a space of holomorphic functions on some domain G of the complex plane $\mathbb{C}$. For a holomorphic self-map φ of G, the *composition operator* C_φ is defined as

$$C_\varphi f = f \circ \varphi, \quad f \in \mathscr{H}.$$

Researchers are interested in the relation between the operator properties of C_φ and the functional properties of the symbol φ. Questions about boundedness, compactness, compact difference, essential norm, cyclicity, complex symmetry and so on, have been considered. These topics are interesting and have received great attention during the past several decades. We refer the reader to the excellent monographs [1–3] and references therein for more detailed information.

Many studies have been done on composition operators on various different spaces of holomorphic functions on the unit disk or the unit ball, such as Hardy spaces, Bergman spaces, Dirichlet spaces, spaces of bounded holomorphic function and weighted Banach spaces with sup-norm (see, e.g., [4–9]).

Recently, in [10] a general class of weighted Hardy spaces of entire functions, which contains the Fock and Fock-type spaces as well as several other well-known spaces, has been considered and bounded composition operators on these spaces have been studied. It is interesting that the characterization for boundedness of such operators depends closely on the behavior of the weight sequence.

The aim of this paper is to introduce a general class of weighted spaces of holomorphic functions in the unit disk and study two special forms of bounded composition operators on these spaces. This newly introduced class of spaces includes classical Hardy, Bergman and Dirichlet spaces.

So far as we know, this class of spaces has not been treated before. We hope that our approach may inspire the reader to investigate further properties and obtain more general results in the future.

The structure of the paper is as follows. In section 2, the class of weighted spaces $\mathscr{H}(\beta)$ of holomorphic functions in the unit disk is performed and a short but essential comparison with the case $\mathscr{H}(\beta, E)$ of entire functions is discussed. Section 3 deals with boundedness of composition operators C_φ induced by affine symbols φ, while in Section 4, the monomial symbols are of our interest.

2. Preliminaries

In 1974 Shields [11] introduced the following weighted spaces. Let a domain G contain the origin and $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. The weighted space $\mathscr{H}(\beta)$ is defined as follows:

$$\mathscr{H}(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{O}(G) : \|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\},$$

where $\mathscr{O}(G)$ is the space of all holomorphic functions on G. This space $\mathscr{H}(\beta)$ is an inner product space with

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2,$$

for every $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Note that $\mathscr{H}(\beta)$ is not necessarily a complete normed space.

Denote by r the radius of the smallest open disk centered at the origin which contains G. It is well known that $\mathscr{H}(\beta)$ is a Hilbert space if and only if $\liminf_{n\to\infty} \beta_n^{1/n} \geqslant r$. In particular, in case $\lim_{n\to\infty} \beta_n^{1/n} = \infty$, we have a Hilbert space of entire functions $\mathscr{H}(\beta, E)$. It is, however, worth to note that since the case of entire functions $\mathscr{H}(\beta, E)$ is essentially different from the one of holomorphic functions $\mathscr{H}(\beta)$ and moreover, it is already considered in [10], we eliminate those weights for which $\lim_{n\to\infty} \beta_n^{1/n} = \infty$.

So throughout this paper, we pay our attention to a model case when G is the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and so the condition

$$1 \leqslant \liminf_{n\to\infty} \beta_n^{1/n} < \infty \tag{1}$$

is supposed to hold.

We note that for $\beta_n = 1$, $\beta_n = (n+1)^{-\frac{1}{2}}$, and $\beta_n = (n+1)^{\frac{1}{2}}$, the space $\mathscr{H}(\beta)$ becomes respectively the classical Hardy space $H^2(\mathbb{D})$, Bergman space $A^2(\mathbb{D})$, and Dirichlet space $\mathscr{D}^2(\mathbb{D})$ in the unit disk.

Note also that $h_n(z) = \beta_n^{-1} z^n$ $(n = 0, 1, 2, \ldots)$ form an orthonormal basis for the Hilbert space $\mathscr{H}(\beta)$.

We can easily verify that $\mathscr{H}(\beta)$ is a functional Hilbert space, and hence C_φ is bounded on $\mathscr{H}(\beta)$ if and only if it maps $\mathscr{H}(\beta)$ into itself.

It is important to have a remark that in the case of weighted spaces of entire functions $\mathscr{H}(\beta, E)$, bounded composition operators C_φ can be induced only by affine functions $\varphi(z) = az + b$. However, in the case of weighted spaces $\mathscr{H}(\beta)$ in the unit disk, the situation is much more complicated. It is not true anymore, as we can see in some examples below.

Example 2.1. For the symbol $\varphi(z) = z^2$, the composition operator C_φ is bounded on each of the classical Hardy, Bergman, and Dirichlet spaces; however C_φ is unbounded in $\mathscr{H}(\beta)$ in the case $\beta_n = \lambda^n$ with $\lambda > 1$.

◁ When $\varphi(z) = z^2$, for every $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\beta)$, we have

$$\|C_\varphi f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2_{H^2(\mathbb{D})},$$

$$\|C_\varphi f\|^2_{A^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{2n+1} \leqslant \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{n+1} = \|f\|^2_{A^2(\mathbb{D})},$$

$$\|C_\varphi f\|^2_{\mathscr{D}^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 (2n+1) \leqslant 2 \sum_{n=0}^{\infty} |a_n|^2 (n+1) = 2\|f\|^2_{\mathscr{D}^2(\mathbb{D})}.$$

For $\beta_n = \lambda^n$ with $\lambda > 1$, we take a function $f(z) = \sum_{n=1}^{\infty} \frac{1}{\lambda^n n} z^n \in \mathscr{O}(\mathbb{D})$. Then

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 = \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n} n^2} \lambda^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

which shows that $f(z) \in \mathscr{H}(\beta)$. However,

$$\|C_\varphi f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_{2n}^2 = \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n} n^2} \lambda^{4n} = \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{n^2},$$

which diverges since $\lambda > 1$. ▷

Remark 2.2. Replacing $\varphi(z) = z^2$ by monomials $\varphi(z) = z^m$ with some integer number $m > 1$, by the same techniques, we can obtain similar results.

3. The Case when $\varphi(z)$ is Affine

In this section, we study boundedness of composition operators on weighted spaces $\mathscr{H}(\beta)$ induced by affine symbols. This study is inspired by the case of entire functions [10].

Let the symbol $\varphi(z) = az + b$ $(a, b \in \mathbb{C})$. For C_φ to be well-defined, $\varphi(z)$ needs to be a self-map of $\mathbb{D}$.

We have the following simple result.

Lemma 3.1. *An affine function $\varphi(z) = az + b$ $(a, b \in \mathbb{C})$ is a self-map of $\mathbb{D}$ if and only if either of the following conditions holds:*

(i) $a = 0$ *and* $|b| < 1$,

(ii) $0 < |a| \leqslant 1$ *and* $|b| \leqslant 1 - |a|$.

$\lhd$ Suppose that $\varphi(z) = az + b$ is a self-map of $\mathbb{D}$. If $a = 0$, then $|\varphi(z)| = |b| < 1$ and hence we get (i). If $a \neq 0$, then $\varphi(\mathbb{D})$ is an open disk centered at the point b of radius $|a|$. Since φ is a self map of $\mathbb{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$. In order for the image disk to be fully contained in the unit disk, we have $|a| \leqslant 1$ and $1 - |b| \geqslant |a|$, which gives (ii).

Conversely, suppose $\varphi(z)$ satisfies (i) or (ii). If (i) holds, then $|\varphi(z)| = |b| < 1$. In case (ii) holds, $|\varphi(z)| = |az + b| \leqslant |az| + |b| < |a| + |b| \leqslant 1$ for all $z \in \mathbb{D}$. In both cases φ is a self-map of $\mathbb{D}$. $\rhd$

From Lemma 3.1, we notice that there are two trivial cases of affine symbols which always induce bounded composition operators. The first case is $\varphi(z) = b$ with $|b| < 1$. The second case is $\varphi(z) = az$ with $0 < |a| \leqslant 1$. In particular, when $|a| = 1$, this is a unitary operator.

As a result, throughout this section, we are interested only in the symbol $\varphi(z) = az + b$ satisfying

$$0 < |a| \leqslant 1 - |b| < 1. \tag{2}$$

With $h_n(z) = \frac{1}{\beta_n} z^n$, we get

$$C_\varphi h_n(z) = \frac{1}{\beta_n}(az + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \frac{z^k}{\beta_n}.$$

Thus,

$$\|C_\varphi h_n\|^2 = \sum_{k=0}^{n} \left(\binom{n}{k} |a|^k |b|^{n-k} \frac{\beta_k}{\beta_n} \right)^2. \tag{3}$$

and

$$\langle C_\varphi h_n, h_k \rangle = \begin{cases} \binom{n}{k} a^k b^{n-k} \frac{\beta_k}{\beta_n}, & n \geqslant k; \\ 0, & k > n. \end{cases} \tag{4}$$

In the sequel, we consider separately necessary conditions and sufficient conditions for boundedness of C_φ on $\mathscr{H}(\beta)$. As it will be seen, several of necessary conditions are "very close" to sufficient ones. We note that some proofs are similar to those of the case of entire functions in [10].

3.1. Necessary conditions. We have following necessary conditions for boundedness of C_φ on $\mathscr{H}(\beta)$, which are similar to those appearing in [10] and proved analogously.

Proposition 3.2. *Let* $\varphi(z) = az + b$ *satisfy* (2). *Consider three statements:*

(i) C_φ *is a bounded composition operator on* $\mathscr{H}(\beta)$.

(ii) $\{\|C_\varphi h_n\|\}_{n \geqslant 0}$ *is a bounded sequence.*

(iii) *The following two inequalities hold:*

$$|a| \limsup_{k \to \infty} \left(\sup_{m \geqslant 0} \left\{ \binom{k+m}{k} |b|^m \frac{\beta_k}{\beta_{k+m}} \right\}^{\frac{1}{k}} \right) \leqslant 1, \tag{5}$$

and

$$|b| \limsup_{m \to \infty} \left(\sup_{k \geqslant 0} \left\{ \binom{k+m}{k} |a|^k \frac{\beta_k}{\beta_{k+m}} \right\}^{\frac{1}{m}} \right) \leqslant 1. \tag{6}$$

Then the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) *hold.*

$\lhd$ (i) $\Longrightarrow$ (ii). Obviously,

$$\|C_\varphi h_n\| \leqslant \|C_\varphi\|_{\text{op}} \quad \text{for all } n \geqslant 0,$$

where $\|C_\varphi\|_{\text{op}}$ is the operator norm of C_φ.

(ii) $\Longrightarrow$ (iii). By (4) and the Cauchy–Schwarz inequality,

$$\binom{k+m}{k}|a|^k|b|^m\frac{\beta_k}{\beta_{k+m}} = |\langle C_\varphi h_{k+m}, h_k\rangle| \leqslant \|C_\varphi h_{k+m}\|$$

for all $k, m \geqslant 0$. From this inequality and boundedness of $\{\|C_\varphi h_n\|\}_{n\geqslant 0}$, both (5) and (6) follow. $\triangleright$

3.2. Sufficient Conditions. The following lemma is adapted from [10, Lemmas 12 and 13] for the case of entire functions. We sketch the proof for the sake of completeness.

Lemma 3.3. *Let $\varphi(z) = az + b$ satisfy* (2). *Suppose that the sequence $\{\|C_\varphi h_n\|\}_{n\geqslant 0}$ is bounded by some $M > 0$. Then there exists an integer $s \in \mathbb{N}$ such that*

$$\sum_{k=0}^{\infty}\sum_{n=(s+1)k}^{\infty} |\langle C_\varphi h_n, h_k\rangle|^2 < \infty.$$

$\triangleleft$ First, let $n \geqslant k \geqslant 0$. By (3) and (4), we have

$$\begin{aligned}(1+|a|)^{n-k}|a|^k|\langle C_\varphi h_n, h_k\rangle| &= \sum_{\ell=0}^{n-k}\binom{n-k}{\ell}|a|^\ell|a|^k\binom{n}{k}|a|^k|b|^{n-k}\frac{\beta_k}{\beta_n}\\ &= \sum_{\ell=0}^{n-k}\binom{k+\ell}{\ell}|a|^k|b|^\ell\frac{\beta_k}{\beta_{k+\ell}}\binom{n}{k+\ell}|a|^{k+\ell}|b|^{n-k-\ell}\frac{\beta_{k+\ell}}{\beta_n}\\ &\leqslant \sum_{\ell=0}^{n-k}\|C_\varphi h_{k+\ell}\|\|C_\varphi h_n\| \leqslant (n-k+1)M^2.\end{aligned}$$

This implies that

$$|\langle C_\varphi h_n, h_k\rangle| \leqslant \frac{(n-k+1)M^2}{(1+|a|)^{n-k}|a|^k} \quad \text{for all } n \geqslant k \geqslant 0.$$

Next, let $s \in \mathbb{N}$ satisfy $(1+|a|)^s|a| > 1$. Then putting $\ell = n-(s+1)k$ and using the above inequality, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=(s+1)k}^{\infty}|\langle C_\varphi h_n, h_k\rangle|^2 &\leqslant M^4\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(sk+\ell+1)^2}{(1+|a|)^{2(sk+\ell)}|a|^{2k}}\\ &\leqslant M^4\sum_{k=0}^{\infty}\frac{(sk+1)^2}{\{(1+|a|)^s|a|\}^{2k}}\sum_{\ell=0}^{\infty}\frac{(\ell+1)^2}{(1+|a|)^{2\ell}},\end{aligned}$$

where the first series converges by the choice of s, while the convergence of the second series is obvious. $\triangleright$

Now we are ready to state and prove sufficient conditions for boundedness of C_φ on $\mathscr{H}(\beta)$.

Proposition 3.4. *Let $\varphi(z) = az + b$ satisfy* (2). *Suppose the following condition holds*

$$\sup_{n\geqslant 1}\left(\sum_{k=1}^{n} k|\langle C_\varphi h_n, h_k\rangle|^2\right) < \infty. \tag{7}$$

Then C_φ is bounded on $\mathscr{H}(\beta)$.

◁ By (7), there is a number $M > 0$ such that

$$\sum_{k=1}^{n} k|\langle C_\varphi h_n, h_k\rangle|^2 \leqslant M^2 \quad \text{for every } n \geqslant 1.$$

Then the sequence $\{\|C_\varphi h_n\|\}_{n\geqslant 0}$ is also bounded by M, because

$$\|C_\varphi h_n\|^2 = \sum_{k=0}^{n} |\langle C_\varphi h_n, h_k\rangle|^2.$$

Hence, by Lemma 3.3, there exists a number $s \in \mathbb{N}$ such that

$$A = \sum_{k=0}^{\infty} \sum_{n=(s+1)k}^{\infty} |\langle C_\varphi h_n, h_k\rangle|^2 < \infty.$$

Take and fix an arbitrary function $f(z) = \sum_{n=0}^{\infty} c_n h_n(z)$ in $\mathscr{H}(\beta)$. First we compute $\langle C_\varphi f, h_k\rangle$ in two cases with respect to k.

If $k \geqslant 1$, then, by (4),

$$\langle C_\varphi f, h_k\rangle = \sum_{n=k}^{\infty} c_n \langle C_\varphi h_n, h_k\rangle = \sum_{n=k}^{(s+1)k-1} c_n \langle C_\varphi h_n, h_k\rangle + \sum_{n=(s+1)k}^{\infty} c_n \langle C_\varphi h_n, h_k\rangle,$$

which implies that

$$\begin{aligned}\frac{1}{2}|\langle C_\varphi f, h_k\rangle|^2 &\leqslant \left(\sum_{n=k}^{(s+1)k-1} |c_n \langle C_\varphi h_n, h_k\rangle|\right)^2 + \left(\sum_{n=(s+1)k}^{\infty} |c_n \langle C_\varphi h_n, h_k\rangle|\right)^2 \\ &\leqslant sk \sum_{n=k}^{(s+1)k-1} |c_n \langle C_\varphi h_n, h_k\rangle|^2 + \left(\sum_{n=(s+1)k}^{\infty} |c_n|^2\right)\left(\sum_{n=(s+1)k}^{\infty} |\langle C_\varphi h_n, h_k\rangle|^2\right) \\ &\leqslant sk \sum_{n=k}^{(s+1)k-1} |c_n \langle C_\varphi h_n, h_k\rangle|^2 + \|f\|^2 \left(\sum_{n=(s+1)k}^{\infty} |\langle C_\varphi h_n, h_k\rangle|^2\right).\end{aligned}$$

If $k = 0$, then $\langle C_\varphi f, h_0\rangle = \sum_{n=0}^{\infty} c_n \langle C_\varphi h_n, h_k\rangle = \sum_{n=(s+1)k}^{\infty} c_n \langle C_\varphi h_n, h_k\rangle$, which implies that

$$\frac{1}{2}|\langle C_\varphi f, h_k\rangle|^2 \leqslant \|f\|^2 \left(\sum_{n=(s+1)k}^{\infty} |\langle C_\varphi h_n, h_k\rangle|^2\right).$$

Next taking summation over $k \geqslant 0$ and using the calculations above, we obtain

$$\begin{aligned}\frac{1}{2}\|C_\varphi f\|^2 &\leqslant s\sum_{k=1}^{\infty} k \sum_{n=k}^{(s+1)k-1} |c_n \langle C_\varphi h_n, h_k\rangle|^2 + \|f\|^2 \sum_{k=0}^{\infty} \left(\sum_{n=(s+1)k}^{\infty} |\langle C_\varphi h_n, h_k\rangle|^2\right) \\ &\leqslant s\sum_{n=1}^{\infty} |c_n|^2 \sum_{k=1}^{n} k|\langle C_\varphi h_n, h_k\rangle|^2 + A\|f\|^2 \leqslant sM^2\|f\|^2 + A\|f\|^2.\end{aligned}$$

From this we conclude that C_φ is bounded on $\mathscr{H}(\beta)$. ▷

We can modify sufficient condition (7) by a slightly stronger condition.

Corollary 3.5. *In Proposition* 3.4, *condition* (7) *can be replaced by the following: There exists a function* $h\colon \{0,1,2,\dots\} \to \mathbb{R}$ *such that*

$$\sum_{k=1}^{\infty} kh^2(k) < \infty \quad \text{and} \quad \sup_{m\geqslant 0}\left\{\binom{k+m}{k}|a|^k|b|^m\frac{\beta_k}{\beta_{k+m}}\right\} \leqslant h(k), \tag{8}$$

for every $k \geqslant 0$.

◁ For any $n \geqslant 1$, by (4), we have

$$\sum_{k=1}^{n} k|\langle C_\varphi h_n, h_k\rangle|^2 = \sum_{k=1}^{n} k\left(\binom{n}{k}|a|^k|b|^{n-k}\frac{\beta_k}{\beta_n}\right)^2 \leqslant \sum_{k=1}^{n} kh^2(k) < \infty,$$

which implies (7). Thus, by Proposition 3.4, C_φ is bounded on $\mathscr{H}(\beta)$. ▷

We can further develop condition (8).

Corollary 3.6. *In Corollary* 3.5, *condition* (8) *can be further replaced by the following inequality*

$$|a|\limsup_{k\to\infty}\left(\sup_{m\geqslant 0}\left\{\binom{k+m}{k}|b|^m\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{k}}\right) < 1, \tag{9}$$

which is very closed to necessary condition (5) *in Proposition* 3.2.

◁ Suppose condition (9) is satisfied. Then there exists $r \in (0,1)$ such that

$$|a|\limsup_{k\to\infty}\left(\sup_{m\geqslant 0}\left\{\binom{k+m}{k}|b|^m\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{k}}\right) < r < 1.$$

For this r, by the properties of lim sup, we can find a constant $M > 0$ (depending only on r) so that

$$\sup_{m\geqslant 0}\left\{\binom{k+m}{k}|a|^k|b|^m\frac{\beta_k}{\beta_{k+m}}\right\} \leqslant Mr^k \quad \text{for all } k \geqslant 0.$$

Taking $h(k) = Mr^k$, we can easily verify that $\sum_{k=1}^{\infty} kh^2(k) < \infty$. Thus we obtain condition (8) and so C_φ is bounded on $\mathscr{H}(\beta)$. ▷

Corollary 3.6 shows that the strict version of (5) is a sufficient condition for boundedness of C_φ on $\mathscr{H}(\beta)$. Using some ideas in the proof of [10, Lemma 18], we prove that the same result is true for the strict version of (6).

Proposition 3.7. *Let* $\varphi(z) = az + b$ *satisfy* (2) *such that*

$$|b|\limsup_{m\to\infty}\left(\sup_{k\geqslant 0}\left\{\binom{k+m}{k}|a|^k\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{m}}\right) < 1. \tag{10}$$

Then C_φ *is bounded on* $\mathscr{H}(\beta)$.

◁ Similarly to the proof of Corollary 3.6, there exist $M > 0$ and $r \in (0,1)$ such that

$$\sup_{k\geqslant 0}\left\{\binom{k+m}{k}|a|^k|b|^m\frac{\beta_k}{\beta_{k+m}}\right\} \leqslant Mr^m \quad \text{for all } m \geqslant 0.$$

From this and (4), it follows that

$$|\langle C_\varphi h_{k+m}, h_k\rangle| = \binom{k+m}{k}|a|^k|b|^m\frac{\beta_k}{\beta_{k+m}} \leqslant Mr^m \quad \text{for all } m,k \geqslant 0. \tag{11}$$

Take and fix an arbitrary polynomial $f(z) = \sum_{n=0}^{\infty} c_n h_n(z) \in \mathscr{H}(\beta)$, where only finite many c_n are nonzero. By (4), we get

$$C_\varphi f = \sum_{k=0}^{\infty}\langle C_\varphi f, h_k\rangle h_k = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{k+m}\langle C_\varphi h_{k+m}, h_k\rangle h_k = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} c_{k+m}\langle C_\varphi h_{k+m}, h_k\rangle h_k.$$

This and (11) imply that

$$\begin{aligned}\|C_\varphi f\| &\leqslant \sum_{m=0}^{\infty}\left\|\sum_{k=0}^{\infty} c_{k+m}\langle C_\varphi h_{k+m}, h_k\rangle h_k\right\| \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}|c_{k+m}|^2|\langle C_\varphi h_{k+m}, h_k\rangle|^2\right)^{\frac{1}{2}} \leqslant M\|f\|\sum_{m=0}^{\infty} r^m = \frac{M}{1-r}\|f\|.\end{aligned}$$

From this and the density of the set of all polynomials in $\mathscr{H}(\beta)$, the assertion follows. ▷

Combining Proposition 3.2, Corollary 3.6 and Proposition 3.7, we have the following important summary for boundedness of C_φ on $\mathscr{H}(\beta)$.

Theorem 3.8. *Let $\varphi(z) = az + b$ satisfy* (2).

(1) *If the operator C_φ is bounded on $\mathscr{H}(\beta)$, then the following two inequalities hold:*

(i) $|a|\limsup_{k\to\infty}\left(\sup_{m\geqslant 0}\left\{\binom{k+m}{k}|b|^m\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{k}}\right) \leqslant 1,$

(ii) $|b|\limsup_{m\to\infty}\left(\sup_{k\geqslant 0}\left\{\binom{k+m}{k}|a|^k\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{m}}\right) \leqslant 1.$

(2) *Conversely, if one of the following conditions hold:*

(iii) $|a|\limsup_{k\to\infty}\left(\sup_{m\geqslant 0}\left\{\binom{k+m}{k}|b|^m\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{k}}\right) < 1,$

(iv) $|b|\limsup_{m\to\infty}\left(\sup_{k\geqslant 0}\left\{\binom{k+m}{k}|a|^k\frac{\beta_k}{\beta_{k+m}}\right\}^{\frac{1}{m}}\right) < 1,$

then the operator C_φ is bounded on $\mathscr{H}(\beta)$.

4. The Case when $\varphi(z)$ are Monomials

4.1. Monomial symbols. Let the symbol $\varphi(z) = cz^m$ ($c \in \mathbb{C}\setminus\{0\}$, $m \in \mathbb{N}\setminus\{1\}$). As in the previous section, to define well C_φ the symbol should be a self-map of $\mathbb{D}$. We can easily check that it is so if and only if $0 < |c| \leqslant 1$, i. e. $c \in \overline{\mathbb{D}}\setminus\{0\}$. Thus throughout this section, such a symbol φ is supposed to consider.

We can easily prove the following criterion for boundedness of C_φ on $\mathscr{H}(\beta)$.

Theorem 4.1. *Let $c \in \overline{\mathbb{D}}\setminus\{0\}$ and $m \in \mathbb{N}\setminus\{1\}$. The operator C_φ induced by the symbol $\varphi(z) = cz^m$ is bounded on $\mathscr{H}(\beta)$ if and only if*

$$\sup_{n\in\mathbb{N}}\frac{|c|^n\beta_{mn}}{\beta_n} < \infty. \tag{12}$$

Moreover, in this case, the above supremum is equal to $\|C_\varphi\|_{\mathrm{op}}$.

◁ Suppose that C_φ is bounded on $\mathscr{H}(\beta)$. Then for every $n \in \mathbb{N}$,

$$\|C_\varphi\|_{\mathrm{op}} \geqslant \|C_\varphi h_n\| = \frac{1}{\beta_n}\|(cz^m)^n\| = \frac{|c|^n \beta_{mn}}{\beta_n},$$

which gives

$$\sup_{n\in\mathbb{N}} \frac{|c|^n \beta_{mn}}{\beta_n} \leqslant \|C_\varphi\|_{\mathrm{op}}.$$

Conversely, suppose that (12) holds. Then for every function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\beta)$, we have

$$\begin{aligned}\|C_\varphi f\|^2 =& \|f(cz^m)\|^2 = \sum_{n=0}^{\infty} |a_n|^2 |c|^{2n} \beta_{mn}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \left(\frac{|c|^n \beta_{mn}}{\beta_n}\right)^2 \\ &\leqslant \left(\sup_{n\in\mathbb{N}} \frac{|c|^n \beta_{mn}}{\beta_n}\right)^2 \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 = \left(\sup_{n\in\mathbb{N}} \frac{|c|^n \beta_{mn}}{\beta_n}\right)^2 \|f\|^2.\end{aligned}$$

This shows that C_φ is bounded on $\mathscr{H}(\beta)$ and also

$$\|C_\varphi\|_{\mathrm{op}} \leqslant \sup_{n\in\mathbb{N}} \frac{|c|^n \beta_{mn}}{\beta_n}.$$

From both necessity and sufficiency it follows that

$$\sup_{n\in\mathbb{N}} \frac{|c|^n \beta_{mn}}{\beta_n} = \|C_\varphi\|_{\mathrm{op}}. \ ▷$$

The following is a consequence of Theorem 4.1.

Corollary 4.2. *Let* $c \in \overline{\mathbb{D}} \setminus \{0\}$ *and* $m \in \mathbb{N} \setminus \{1\}$. *If*

$$\limsup_{n\to\infty} \left(\frac{\beta_{mn}}{\beta_n}\right)^{\frac{1}{n}} > \frac{1}{|c|},$$

then C_{cz^m} *is unbounded on* $\mathscr{H}(\beta)$. *In particular, if*

$$\limsup_{n\to\infty} \left(\frac{\beta_{mn}}{\beta_n}\right)^{\frac{1}{n}} = \infty,$$

then C_{cz^m} *is unbounded on* $\mathscr{H}(\beta)$ *for every* $0 < |c| \leqslant 1$.

◁ By the hypothesis, there exist a sequence $(n_k) \uparrow \infty$ and a number $\alpha > 0$ such that

$$\left(\frac{\beta_{mn_k}}{\beta_{n_k}}\right)^{\frac{1}{n_k}} > \alpha > \frac{1}{|c|} \quad \text{for every } k \in \mathbb{N}.$$

Hence,

$$\frac{|c|^{n_k} \beta_{mn_k}}{\beta_{n_k}} > (\alpha|c|)^{n_k} \to \infty \ \text{ as } \ k \to \infty.$$

From this and Theorem 4.1 the assertion follows. ▷

4.2. Some Notes on a General Symbol. In view of Corollary 4.2, we see that the situation for composition operators C_φ on general weighted spaces $\mathscr{H}(\beta)$ is much more complicated than on classical Hardy space $H^2(\mathbb{D})$ and Bergman space $A^2(\mathbb{D})$. More precisely, it is well-known that every holomorphic self-map of $\mathbb{D}$ with relatively compact image in $\mathbb{D}$ induces a bounded composition operator on these spaces. However, by Corollary 4.2, this is not valid for general weighted spaces $\mathscr{H}(\beta)$.

We end this section with the following sufficient conditions for boundedness of the operators C_φ induced by such holomorphic self-maps of $\mathbb{D}$.

Theorem 4.3. *Let φ be a holomorphic self-map of $\mathbb{D}$ such that $\overline{\varphi(\mathbb{D})}$ is compact in $\mathbb{D}$. If*

$$\sum_{n=0}^{\infty} \beta_n^2 < \infty, \tag{13}$$

then C_φ is bounded on $\mathscr{H}(\beta)$.

$\triangleleft$ Take a number $r \in (0,1)$ so that $\varphi(\mathbb{D}) \subset r\mathbb{D}$. Then, for every $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\beta)$, we have

$$\sup_{|z|\leqslant r} |f(z)| \leqslant \sum_{n=0}^{\infty} |a_n| r^n \leqslant \left(\sum_{n=0}^{\infty} \frac{r^{2n}}{\beta_n^2}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2\right)^{\frac{1}{2}} = C_r \|f\|,$$

where

$$C_r := \left(\sum_{n=0}^{\infty} \frac{r^{2n}}{\beta_n^2}\right)^{\frac{1}{2}} < \infty, \text{ since, by (1), } \limsup_{n\to\infty} \frac{r^2}{\beta_n^{\frac{2}{n}}} \leqslant r^2 < 1.$$

Now writing $C_\varphi f(z) = \sum_{n=0}^{\infty} b_n z^n$ and using the Cauchy integral formula, for every $\delta \in (0,1)$, we get

$$|b_n| = \frac{|(C_\varphi f)^n(0)|}{n!} \leqslant \frac{1}{2\pi} \left| \int\limits_{|z|=\delta} \frac{C_\varphi f(z)}{z^{n+1}} dz \right| \leqslant \frac{1}{\delta^n} \max_{|z|=\delta} |C_\varphi f(z)| \leqslant \frac{1}{\delta^n} \sup_{|z|\leqslant r} |f(z)| \leqslant \frac{C_r}{\delta^n} \|f\|,$$

which implies that $|b_n| \leqslant C_r \|f\|$ for every $n \in \mathbb{N}$.

Consequently,

$$\|C_\varphi f\| = \left(\sum_{n=0}^{\infty} |b_n|^2 \beta_n^2\right)^{\frac{1}{2}} \leqslant C_r \|f\| \left(\sum_{n=0}^{\infty} \beta_n^2\right)^{\frac{1}{2}}.$$

From this the assertion follows. $\triangleright$

Remark 4.4. Condition (13) is essential but not sharp for Theorem 4.3. Indeed, for

$$\beta_n = \begin{cases} 1, & n = 0, \\ 1, & n \equiv 1, 2, 3 \mod 4, \\ n^n, & n \equiv 0 \mod 4,\ n \neq 0, \end{cases}$$

(13) does not hold and, by Corollary 4.2, C_{cz^m} is unbounded on $\mathscr{H}(\beta)$ for every $c \in \overline{\mathbb{D}} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{1\}$.

On the other hand, for $\beta_n = 1$, (13) also does not hold, however in this case $\mathscr{H}(\beta)$ is Hardy space $H^2(\mathbb{D})$ and hence C_φ is bounded on $\mathscr{H}(\beta)$ for every holomorphic function φ with $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$.

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ОГРАНИЧЕННЫЕ КОМПОЗИЦИОННЫЕ ОПЕРАТОРЫ НА ВЕСОВЫХ ФУНКЦИОНАЛЬНЫХ ПРОСТРАНСТВАХ НА ЕДИНИЧНОМ КРУГЕ

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Аннотация. Введен общий класс весовых пространств $\mathscr{H}(\beta)$ голоморфных функций в единичном круге $\mathbb{D}$, содержащий в качестве частных случаев классические пространства Харди, Бергмана и Дирихла. Полностью охарактеризована ограниченность на этих пространствах $\mathscr{H}(\beta)$ композиционных операторов C_φ, порожденных аффинными и мономиальными символами φ. Установлено также достаточное условие, при выполнении которого оператор C_φ, порожденный символом φ с относительно компактным образом $\varphi(\mathbb{D})$ в $\mathbb{D}$, является ограниченным на $\mathscr{H}(\beta)$. Отменим, что в пространствах $\mathscr{H}(\beta)$ описание ограниченности композиционных операторов C_φ зависит не только от функциональных свойств символов φ, но и от поведения весовой последовательности β.

Ключевые слова: композиционный оператор, весовое пространство, весовая последовательность, голоморфные функции, единичный круг.

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