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COLOR ENERGY OF SOME CLUSTER GRAPHS

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Abstract. Let G be a simple connected graph. The energy of a graph G is defined as sum of the absolute eigenvalues of an adjacency matrix of the graph G. It represents a proper generalization of a formula valid for the total π-electron energy of a conjugated hydrocarbon as calculated by the Huckel molecular orbital (HMO) method in quantum chemistry. A coloring of a graph G is a coloring of its vertices such that no two adjacent vertices share the same color. The minimum number of colors needed for the coloring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$. The color energy of a graph G is defined as the sum of absolute values of the color eigenvalues of G. The graphs with large number of edges are referred as cluster graphs. Cluster graphs are graphs obtained from complete graphs by deleting few edges according to some criteria. It can be obtained on deleting some edges incident on a vertex, deletion of independent edges/triangles/cliques/path P3 etc. Bipartite cluster graphs are obtained by deleting few edges from complete bipartite graphs according to some rule. In this paper, the color energy of cluster graphs and bipartite cluster graphs are studied.

Key words: color adjacency matrix, color eigenvalues, color energy.

Mathematical Subject Classification (2010): 05C15, 05C50.

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1. Introduction

Let G be a simple undirected graph with n vertices. The energy of a graph was defined by I. Gutman [1] in 1978 as sum of the absolute eigenvalues of the graph G. A coloring of a graph G [2] is a coloring of its vertices such that no two adjacent vertices share the same color. The minimum number of colors needed for coloring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$.

In 2013 C. Adiga, E. Sampathkumar, M. A. Sriraj and A. S. Shrikanth [3] have introduced the energy of colored graph. The entries of the color adjacency matrix $A_c(G)$ are as follows: If $c(v_i)$ is the color of vertex v_i, then

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j); \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i)=c(v_j); \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_c(G)$ is denoted by $\phi(A_c(G),\lambda) = \det(\lambda I - A_c(G))$. The roots of characteristic polynomial $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the color eigenvalues of $A_c(G)$. The color energy of a graph G is the sum of absolute values of color eigenvalues of G, i. e., $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$. For more information on energy and color energy of a graph, refer [4–10].

Basically, cluster graphs are graphs obtained from complete graphs by deleting few edges according to some criteria. I. Gutman and L. Pavlović [11] have studied several cluster graphs and found their energies. Bicluster graphs are the byproduct of complete bipartite graphs, obtained by deleting few edges from complete bipartite graphs. H. B. Walikar and H. S. Ramane [12] have studied energy of bipartite cluster graphs. In Section 2 we compute color energy of some cluster graphs. In Section 3 we establish color energy of bipartite cluster graphs.

2. Color Energy of Some Cluster Graphs

Definition 2.1 [13]. Let $(K_m)_i$, $i = 1, 2, \ldots, k$, $1 \leqslant k \leqslant \lfloor \frac{n}{m} \rfloor$, $1 \leqslant m \leqslant n$, be independent complete subgraphs with m vertices of the complete graph K_n, $n \geqslant 3$. The cluster graph $Ka_n(m,k)$ obtained from K_n, by deleting all edges of $(K_m)_i$, $i = 1, 2, \ldots, k$. In addition $Ka_n(m,0) = Ka_n(0,k) = Ka_n(0,0) = K_n$.

Definition 2.2 [13]. For fixed integers $n \geqslant 3$ and $0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, the cluster graph $K_{b_n}(k)$ is obtained from K_n by the deletion of k independent edges.

Definition 2.3 [13]. For fixed integers $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$, the cluster graph $K_{c_n}(m)$ is obtained from K_n by deleting a m-clique.

Definition 2.4 [13]. Let $n \geqslant 3$ and $1 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$ be fixed integers. The cluster graph $K_{f_n}(k)$ is obtained from K_n by the deletion of k disjoint triangles.

Definition 2.5 [13]. For fixed integers n and k, $n \geqslant 3$ and $0 \leqslant k \leqslant n-1$, the cluster graph $K_{a_n}(k)$ is obtained from K_n by the deletion of k edges with a common end vertex.

Definition 2.6 [13]. For $n \geqslant 3$ and $0 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$, the cluster graph $K_{e_n}(k)$ is obtained from K_n by the deletion of k independent paths P_3.

Definition 2.7. For $n \geqslant 4$ and $0 \leqslant k \leqslant \lfloor \frac{n}{4} \rfloor$, the cluster graph $K_{d_n}(k)$ obtained from K_n by the deletion of k independent paths P_4.

Theorem 2.8. *For $n \geqslant 3$, $0 \leqslant k \leqslant \lfloor \frac{n}{m} \rfloor$ and $1 \leqslant m \leqslant n$, $E_c(Ka_n(m,k)) = n - 2m + 2k(m-1) + \sqrt{(4-8k)m^2 + (2k+n-2)4m + (n-2)^2}$.*

$\lhd$ For the cluster graph $Ka_n(m,k)$, chromatic number $\chi(Ka_n(m,k)) = n - (m-1)k$. We have

$$A_c\left(Ka_n(m,k)\right) = \left[\begin{array}{c|c} X_{mk} & J_{mk\times(n-mk)} \\ \hline J_{(n-mk)\times mk} & (J-I)_{(n-mk)} \end{array}\right],$$

$$X = \left[\begin{array}{c|c|c|c} (I-J)_m & J_m & \cdots & J_m \\ \hline J_m & (I-J)_m & \cdots & J_m \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline J_m & J_m & \cdots & (I-J)_m \end{array}\right]_{mk\times mk},$$

where J is the matrix with all entries one. Consider $\det\left(\lambda I - A_c\left(Ka_n(m,k)\right)\right)$.

$$\text{Step 1: Replace } R_i \text{ by } R_i' = \begin{cases} R_i - R_{i+1}, & \text{for } i = 1, 2, \ldots m-1, m+1, m+2, \ldots, \\ & \qquad 2m-1, 2m+1, \ldots, mk-2, mk-1; \\ R_i - R_{i-1}, & \text{for } i = n, n-1, \ldots, mk+3, mk+2. \end{cases}$$

Then, $\det\left(\lambda I - A_c\left(Ka_n(m,k)\right)\right)$ reduces to a new determinant, say

$$\det(C) = \left|\begin{array}{c|c|c|c|c} Z_{m-1\times m} & 0_{m-1\times m} & \cdots & 0_{m-1\times m} & 0_{m-1\times n-mk} \\ \hline Y_{1\times m} & -J_{1\times m} & \cdots & -J_{1\times m} & -J_{1\times n-mk} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0_{m-1\times m} & 0_{m-1\times m} & \cdots & Z_{m-1\times m} & 0_{m-1\times n-mk} \\ \hline -J_{1\times m} & -J_{1\times m} & \cdots & Y_{1\times m} & -J_{1\times n-mk} \\ \hline -J_{1\times m} & -J_{1\times m} & \cdots & -J_{1\times m} & M_{1\times n-mk} \\ \hline 0_{n-3k-1\times m} & 0_{n-3k-1\times m} & \cdots & 0_{n-3k-1\times m} & -Z_{n-3k-1\times n-mk} \end{array}\right|,$$

where $Y = [1\ 1 \ldots 1\ \lambda]$, $M = [\lambda\ -1 \ldots -1\ -1]$ and

$$Z = (\lambda-1)I - (\lambda-1)\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Step 2: In $\det(C)$, replacing C_i by

$$C_i' = \begin{cases} C_i + C_{i-1}, & \text{for } i = 2,3,\ldots,m-1,m,m+2,m+3,\ldots, \\ & \quad 2m, 2m+2, 2m+3, \ldots, mk-1, mk; \\ C_i + C_{i+1}, & \text{for } i = n-1, n-2, \ldots, mk+1, \end{cases}$$

it reduces to

$$\det(C) = \left|\begin{array}{c|c|c|c|c|c|c|c|c} I_{m-1} & 0_{m-1\times 1} & 0_{m-1} & 0_{m-1\times 1} & \cdots & 0_{m-1} & 0_{m-1\times 1} & 0_{m-1\times 1} & 0_{m-1\times q} \\ \hline M & \lambda+m & -M & -m & \cdots & -M & -m & -(n-mk) & N \\ \hline 0_{m-1} & 0_{m-1\times 1} & I_{m-1} & 0_{m-1\times 1} & \cdots & 0_{m-1} & 0_{m-1\times 1} & 0_{m-1\times 1} & 0_{m-1\times q} \\ \hline -M & -m & M & \lambda+m-1 & \cdots & -M & -m & -(n-mk) & N \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline 0_{m-1} & 0_{m-1\times 1} & 0_{m-1} & 0_{m-1\times 1} & \cdots & I_{m-1} & 0_{m-1\times 1} & 0_{m-1\times 1} & 0_{m-1\times q} \\ \hline -M & -m & -M & -m & \cdots & -M & \lambda+m-1 & -(n-mk) & N \\ \hline -M & -m & -M & -m & \cdots & -M & -m & \lambda-q & N \\ \hline 0_{q\times m-1} & 0_{q\times 1} & 0_{q\times m-1} & 0_{q\times 1} & \cdots & 0_{q\times m-1} & 0_{q\times 1} & 0_{q\times 1} & I_q \end{array}\right|,$$

where $q = n-mk-1$, $M_{1\times m-1} = [1\ 2\ m-1]$ and $N_{1\times n-k-1} = [-(n-mk-1)\ldots -3\ -2\ -1]$.

Step 3: On expanding the $\det(C)$ successively along the rows R_i, for $i = 1,2,3,\ldots,k-1, k+1, k+2, \ldots, 2k-1, 2k+1, 2k+2, \ldots, mk-2, mk-1, mk+2, mk+3, \ldots, n-1, n$, it becomes $(\lambda-1)^{(m-1)k}(\lambda+1)^{n-mk-1}\det(D)$, where

$$\det(D) = \begin{vmatrix} \lambda+m-1 & -m & -m & \cdots & -m & -(n-mk) \\ -m & \lambda+m-1 & -m & \cdots & -m & -(n-mk) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -m & -m & -m & \cdots & \lambda+m-1 & -(n-mk) \\ -m & -m & -m & \cdots & -m & \lambda-n+mk+1 \end{vmatrix}_{k+1\times k+1}.$$

Step 4: In $\det(D)$, replacing R_i by $R_i' = R_i - R_{i+1}$, for $i = 1,2,\ldots,k$, we obtain

$$\det(D) = (\lambda+2m-1)^{k-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -m & -2m & \cdots & -(k-1)m & \lambda+(2-k)m-1 & -(n-3k) \\ -m & -2m & \cdots & -(k-1)m & -km & \lambda+mk-n+1 \end{vmatrix}.$$

Step 5: In $\det(D)$, replacing C_i by $C_i' = C_i + C_{i-1} + \cdots + C_1$, for $i = k, k-1, \ldots, 2$, it reduces to

$$\det(D) = (\lambda + 2m - 1)^{k-1} \begin{vmatrix} \lambda + (2-k)m - 1 & mk - n \\ -mk & \lambda + mk - n + 1 \end{vmatrix}$$
$$= (\lambda + 2m - 1)^{k-1}\big(\lambda^2 - (2m-n)\lambda - ((2m-1)(n-1) - (2m^2 - 2m)k)\big).$$

Thus,

$$\phi(A_c(Ka_n(m,k)), \lambda) = (\lambda - 1)^{(m-1)k}(\lambda + 1)^{n-mk-1}$$
$$\times (\lambda + 2m - 1)^{k-1}\big(\lambda^2 - (2m-n)\lambda - ((2m-1)(n-1) - (2m^2 - 2m)k)\big).$$

So, the color spectrum of $Ka_n(m,k)$ is

$$\Big\{1((m-1)k\text{times}), -1(n - mk - 1\text{times}), 1 - 2m(k - 1\text{times}),$$
$$\frac{n - 2m + \sqrt{(4-8k)m^2 + (2k+n-2)4m + (n-2)^2}}{2},$$
$$\frac{n - 2m - \sqrt{(4-8k)m^2 + (2k+n-2)4m + (n-2)^2}}{2}\Big\}.$$

Hence, $E_c(Ka_n(m,k)) = n - 2m + 2k(m-1) + \sqrt{(4-8k)m^2 + (2k+n-2)4m + (n-2)^2}$. ▷

Corollary 2.9. *For $n \geqslant 3$ and $0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, $E_c(K_{b_n}(k)) = n + 2k - 4 + \sqrt{(n+2)^2 - 16k}$.*

◁ Observe that $K_{b_n}(k)$ is a special case of $Ka_n(m,k)$, when $m = 2$. Thus, by substituting $m = 2$ in Theorem 2.8, the result follows. ▷

Corollary 2.10. *For $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$, $E_c(K_{c_n}(m)) = n - 2 + \sqrt{(n-2)^2 - 4m(n-m)}$.*

◁ The proof follows by noting that $K_{c_n}(m) = Ka_n(m,1)$ in Theorem 2.8. ▷

Corollary 2.11. *For $n \geqslant 3$ and $1 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$, $E_c(K_{f_n}(k)) = n + 4k - 6 + \sqrt{(n+4)^2 - 48k}$.*

◁ The proof follows from Theorem 2.8 by noting that $K_{f_n}(k) = Ka_n(3,k)$. ▷

Theorem 2.12. *For $n \geqslant 3$ and $0 \leqslant k \leqslant n-1$,*

$$\phi(A_c(K_{a_n}(k)), \lambda) = (\lambda + 1)^{n-4}(\lambda^4 - (n-4)\lambda^3$$
$$-(3n - k - 5)\lambda^2 - (k(n-k-1) - 2)\lambda + (2+k)(n-k) - 4).$$

◁ Let $v_1, v_2, \ldots, v_n$ be the vertices of complete graph K_n. The cluster graph $K_{a_n}(k)$ obtained from K_n by the deletion of k edges with common end vertex v_i, $i = 1, 2, 3, \ldots, n$, and $\chi(K_{a_n}(k)) = n - 1$. We have

$$A_c(K_{a_n}(k)) = \left[\begin{array}{c|c|c} 0 & C_{1\times k} & J_{1\times n-k-1} \\ \hline C^T_{k\times 1} & (J-I)_k & J_{k\times n-k-1} \\ \hline J_{n-k-1\times 1} & J_{n-k-1\times k} & (J-I)_{n-k-1} \end{array}\right],$$

where $C_{1\times k} = [-1 \quad 0 \quad 0 \ldots 0]$. Consider $\det(\lambda I - A_c(K_{a_n}(k)))$.

Step 1: Replace R_i by $R_i' = R_i - R_{i+1}$, for $i = 2, 3, \ldots, k, k+1, \ldots, n-2, n-1$. Then, $\det(\lambda I - A_c(K_{a_n}(k)))$ reduces to $(\lambda+1)^{n-4}\det(C)$. Where

$$\det(C) = \left|\begin{array}{c|c|c} \lambda & -C_{1\times k} & -J_{1\times n-k-1} \\ \hline C^T_{k\times m} & B_{k-1\times k} & 0_{k-1\times n-k-1} \\ \hline 0_{1\times 1} & Y_{1\times k} & -J_{1\times n-k-1} \\ \hline 0_{n-k-2\times 1} & 0_{n-k-2\times k} & B_{n-k-2\times n-k-1} \\ \hline -J_{1\times 1} & -J_{1\times k} & Y_{1\times n-k-1} \end{array}\right|,$$

where $Y = [-1\ \ -1 \ldots -1\ \lambda]$, $C = [1\ 0\ 0\ 0]$ and

$$B = (\lambda+1)I - (\lambda+1)\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Step 2: In $\det(C)$, replacing C_i by $C_i' = C_i + C_{i-1}$, for $i = 3, 4, 5, \ldots, k+1, k+3, \ldots, n$, we obtain,

$$\det(C) = \left|\begin{array}{c|c|c|c|c} \lambda & J_{1\times k-1} & 1 & M & k-n+1 \\ \hline C^T_{k-1\times 1} & (\lambda+1)I_{k-1} & 0_{k-1\times 1} & 0_{k-1\times n-k-2} & 0_{k-1\times 1} \\ \hline 0 & N & \lambda-k+1 & M & k-n+1 \\ \hline 0_{n-k-2\times 1} & 0_{n-k-2\times k-1} & 0_{n-k-2} & (\lambda+1)I_{n-k-2} & 0_{n-k-2\times 1} \\ \hline -1 & N & -k & M & \lambda-n+k+2 \end{array}\right|,$$

where $M_{1\times n-k-2} = [-1\ \ -2\ \ -3 \ldots -(n-k-2)]$, $N_{1\times k-1} = [-1\ \ -2\ \ -3 \ldots -(k-1)]$.

Step 3: On expanding the $\det(C)$ successively along the rows R_i, for $i = 3, 4, \ldots, k, k+2, \ldots, n-2,\ n-1$, it reduces to

$$\det(C) = \begin{vmatrix} \lambda & 1 & 1 & k-n+1 \\ 1 & \lambda+1 & 0 & 0 \\ 0 & -1 & \lambda-k+1 & k-n+1 \\ -1 & -1 & -k & \lambda-n+k+2 \end{vmatrix}$$

$$= (\lambda^4 - (n-4)\lambda^3 - (3n-k-5)\lambda^2 - (k(n-k-1)-2)\lambda + (2+k)(n-k)-4).$$

Thus,

$$\phi(A_c(K_{a_n}(k)),\lambda) = (\lambda+1)^{n-4}(\lambda^4 - (n-4)\lambda^3$$
$$-(3n-k-5)\lambda^2 - (k(n-k-1)-2)\lambda + (2+k)(n-k)-4).\ \rhd$$

Theorem 2.12. *For $n \geqslant 3$ and $1 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$, $\phi(A_c(K_{e_n}(k)),\lambda) = (\lambda+1)^{n-2k-2}(\lambda^2+2\lambda-4)^{k-1}(\lambda^4 - (n-4)\lambda^3 - (3n-6k-1)\lambda^2 + (2n-2k-6)\lambda + 4(n-2k-1))$.*

$\lhd$ Consider the cluster graph $K_{e_n}(k)$ obtained from K_n by the deletion of k independent paths P_3. Since $\chi(K_{e_n}(k)) = n-k$. We have

$$A_c(K_{e_n}(k)) = \left[\begin{array}{c|c|c|c} (J-I)_k & (J-2I)_k & J_k & J_{k\times(n-3k)} \\ \hline (J-2I)_k & (J-I)_k & (J-I)_k & -J_{k\times(n-3k)} \\ \hline J_k & (J-I)_k & (J-I)_k & J_{k\times(n-3k)} \\ \hline J_{(n-3k)\times k} & -J_{n-3k\times k} & J_{n-3k\times k} & (J-I)_{(n-3k)} \end{array}\right].$$

Consider $\det(\lambda I - A_c(K_{e_n}(k)))$.

Step 1: Replace R_i by

$$R_i' = \begin{cases} R_i - R_{i+1}, & \text{for } i = 1, 2, \ldots, k-1, k+1, k+2, \ldots, 2k-1, \\ & \quad 2k+1, 2k+2, \ldots, 3k-1, \\ R_i - R_{i-1}, & \text{for } i = n, n-1, \ldots, 3k+2. \end{cases}$$

Then, $\det\left(\lambda I - A_c\left(K_{e_n}(k)\right)\right)$ reduces to a new determinant, say

$$\det(C) = \left|\begin{array}{c|c|c|c} X_{k-1\times k} & Y_{k-1\times k} & 0_{k-1\times k} & 0_{k-1\times(n-3k)} \\ \hline P_{1\times k} & M_{1\times k} & -J_{1\times k} & -J_{1\times(n-3k)} \\ \hline Y_{k-1\times k} & X_{k-1\times k} & Z_{k-1\times k} & 0_{k-1\times(n-3k)} \\ \hline M_{1\times k} & P_{1\times k} & Q_{1\times k} & -J_{1\times(n-3k)} \\ \hline 0_{k-1\times k} & Z_{k-1\times k} & X_{k-1\times k} & 0_{k-1\times(n-3k)} \\ \hline -J_{1\times k} & Q_{1\times k} & P_{1\times k} & -J_{1\times(n-3k)} \\ \hline -J_{1\times k} & -J_{1\times k} & -J_{1\times k} & R_{1\times(n-3k)} \\ \hline 0_{n-3k-1\times k} & 0_{n-3k-1\times k} & 0_{n-3k-1\times k} & X_{n-3k-1\times(n-3k)} \end{array}\right|,$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$X = (\lambda+1)I - (\lambda+1)B$, $\quad Y = 2I - 2B$ and $Z = I - B$. Also $P = [-1 \;\; -1 \ldots -1 \;\; \lambda]$, $R = [\lambda \;\; -1 \ldots -1 \;\; -1]$, $M = [-1 \;\; -1 \ldots -1 \;\; 1]$ and $Q = [-1 \;\; -1 \ldots -1 \;\; 0]$.

Step 2: In $\det(C)$, replacing C_i by

$$C_i' = \begin{cases} C_i + C_{i+1}, & \text{for } i = n-1, n-2, \ldots, 3k+1; \\ C_i + C_{i-1} + \ldots + C_1, & \text{for } i = k, k-1, \ldots, 2; \\ C_i + C_{i-1} + \ldots + C_{k+1}, & \text{for } i = 2k, 2k-1, \ldots, k+2; \\ C_i + C_{i-1} + \ldots + C_{2k+1}, & \text{for } i = 3k, 3k-1, \ldots, 2k+2. \end{cases}$$

We obtain

$$\det(C)$$

$$= \left|\begin{array}{c|c|c|c|c|c|c|c} (\lambda+1)I_{k-1} & 0_{k-1\times1} & 2I_{k-1} & 0_{k-1\times1} & 0_{k-1} & 0_{k-1\times1} & 0_{k-1\times1} & 0_{k-1\times q} \\ \hline X & \lambda-k+1 & X & -k+2 & X & -k & -(n-3k) & Y \\ \hline 2I_{k-1} & 0_{k-1\times1} & (\lambda+1)I_{k-1} & 0_{k-1} & 0_{k-1} & 0_{k-1\times1} & 0_{k-1\times1} & 0_{k-1\times q} \\ \hline X & -k+2 & X & \lambda-k+1 & X & -k+1 & -(n-3k) & Y \\ \hline 0_{k-1} & 0_{k-1\times1} & I_{k-1} & 0_{k-1\times1} & (\lambda+1)I_{k-1} & 0_{k-1\times1} & 0_{k-1\times1} & 0_{k-1\times q} \\ \hline X & -k & X & -k+1 & X & \lambda-k+1 & -(n-3k) & Y \\ \hline X & -k & X & -k & X & -k & \lambda+q & Y \\ \hline 0_{k-1} & 0_{k-1\times1} & 0_{k-1} & 0_{k-1\times1} & 0_{k-1\times1} & 0_{k-1\times1} & 0_{k-1\times1} & (\lambda+1)I_{k-1\times q} \end{array}\right|,$$

where $q = n-3k-1$, $X_{1\times k-1} = [-1 \;\; -2 \;\; -3 \ldots -(k-1)]$ and $Y_{1\times n-3k-1} = [-(n-3k-1) \;\; -3 \;\; -2 \;\; -1]$.

Step 3: In $\det(C)$, replacing C_i by $C_i' = (\lambda+1)C_i - 2C_j$, for $i = k+1, k+2, \ldots, 2k-1$ and $j = 1, 2, \ldots, k-1$ and expanding $\det(C)$ successively along the rows R_i, for $i = 1, 2, \ldots, k-1$,

$3k+2, 3k+3\ldots, n-1, n$, it becomes $(\lambda+1)^{n-3k-1}\det(D)$, where

$$\det(D)$$

$$=\begin{vmatrix} \lambda-k+1 & -(\lambda-1) & \cdots & -(k-1)(\lambda-1) & -(k-2) & -1 & \cdots & -k & -(n-3k)\\ 0 & \lambda^2+2\lambda-3 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda^2+2\lambda-3 & 0 & 0 & \cdots & 0 & 0\\ -(k-2) & -(\lambda-1) & \cdots & -(k-1)(\lambda-1) & \lambda-k+1 & -1 & \cdots & -(k-1) & -(n-3k)\\ 0 & \lambda+1 & \cdots & 0 & 0 & \lambda+1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda+1 & 0 & 0 & \cdots & 0 & 0\\ -k & -(\lambda-1) & \cdots & -(k-1)(\lambda-1) & -(k-1) & -1 & \cdots & \lambda-k+1 & -(n-3k)\\ -k & -(\lambda-1) & \cdots & -(k-1)(\lambda-1) & -k & -1 & \cdots & -k & \lambda-n+3k+1 \end{vmatrix}.$$

Step 4: In $\det(D)$, replacing C_i by $C_i' = C_i - C_{k+i}$, for $i = 2, 3, \ldots, k-1$ and then expanding $\det(F)$ successively along the rows R_i, for $i = 2, 3, \ldots, k, k+2, k+3\ldots, 2k-1, 2k$, it reduces to

$$\det(D) = (\lambda+1)^{k-1}(\lambda^2+2\lambda-4)^{k-1}\begin{vmatrix} \lambda-k+1 & 2-k & -k & 3k-n\\ 2-k & \lambda-k+1 & 1-k & 3k-n\\ -k & 1-k & \lambda-k+1 & 3k-n\\ -k & -k & -k & \lambda-n+3k+1 \end{vmatrix}$$
$$= (\lambda+1)^{k-1}(\lambda^2+2\lambda-4)^{k-1}(\lambda^4-(n-4)\lambda^3$$
$$-(3n-6k-1)\lambda^2+(2n-2k-6)\lambda+4(n-2k-1)).$$

Thus, $\phi(A_c(K_{e_n}(k)), \lambda) = (\lambda+1)^{n-2k-2}(\lambda^2+2\lambda-4)^{k-1}(\lambda^4-(n-4)\lambda^3-(3n-6k-1)\lambda^2+(2n-2k-6)\lambda+4(n-2k-1))$. $\triangleright$

Theorem 2.13. *For $n \geqslant 4$ and $1 \leqslant k \leqslant \lfloor \frac{n}{4} \rfloor$, $\phi(A_c(K_{d_n}(k)), \lambda) = (\lambda+1)^{n-4k-1}(\lambda^2+3\lambda-2)^{k-1}(\lambda^2+\lambda-4)^k(\lambda^3-(n-4)\lambda^2-(3n-10k-1)\lambda+(2n-6k-2)$.*

$\triangleleft$ Consider the cluster graph $K_{d_n}(k)$ obtained from complete graph K_n by the deletion of k independent paths P_4. Since $\chi(K_{d_n}(k)) = n-2k$. We have

$$A_c\left(K_{d_n}(k)\right) = \left[\begin{array}{c|c|c} (J-I)_{2k} & (J-2I)_{2k} & J_{2k\times(n-4k)}\\ \hline (J-2I)_{2k} & X_{2k} & J_{2k\times(n-4k)}\\ \hline J_{(n-4k)\times 2k} & J_{(n-4k)\times 2k} & (J-I)_{(n-4k)} \end{array}\right],$$

$$X = \left[\begin{array}{c|c|c|c} 0_{2\times2} & J_{2\times2} & \cdots & J_{2\times2}\\ \hline J_{2\times2} & 0_{2\times2} & \cdots & J_{2\times2}\\ \hline \vdots & \vdots & \ddots & \vdots\\ \hline J_{2\times2} & J_{2\times2} & \cdots & 0_{2\times2} \end{array}\right]_{2k\times2k}.$$

Consider $\det\left(\lambda I - A_c\left(K_{d_n}(k)\right)\right)$.

Step 1: Replace R_i by

$$R_i' = \begin{cases} R_i - R_{i+1}, & \text{for } i = 1, 2, \ldots, 2k-1, 2k+1, 2k+3, \ldots, 4k-3, 4k-1;\\ R_i - R_{i-1}, & \text{for } i = n, n-1, \ldots, 4k+2. \end{cases}$$

Then, $\det\left(\lambda I - A_c\left(K_{d_n}(k)\right)\right)$ reduces to a new determinant, say $\det(C)$.

Step 2: In $\det(C)$, replacing C_i by

$$C_i' = \begin{cases} C_i + C_{i+1}, & \text{for } i = n-1, n-2, \ldots, 4k+2, 4k+1; \\ C_i + C_{i-1} + \ldots + C_1, & \text{for } i = 2k, 2k-1, \ldots, 2; \\ C_i + C_{i-1} + \ldots + C_{2k}, & \text{for } i = 4k, 4k-1, \ldots, 2k+2, \end{cases}$$

a new determinant, $\det(D)$ is obtained.

Step 3: In $\det(D)$, replace C_i by $C_i' = (\lambda+1)C_i - 2C_j$, for $i = 2k+1, 2k+2, \ldots, 4k-1$ and $j = 1, 2, \ldots, 2k-1$. It reduces to $\det(E)$.

Step 4: On expanding $\det(E)$ successively along the rows R_i, for $i = 1, 2, \ldots, 2k-1, 2k+1, 2k+3, \ldots, 4k-3, 4k-1, 4k+2, 4k+3, \ldots, n-1, n$, simplifies $(\lambda+1)^{n-4k+1}(\lambda^2+\lambda-4)^k \det(F)$ of order $k+2$, which is shown as follows

$$\det(F) = \begin{vmatrix} \lambda - 2k + 1 & -2(\lambda-1) & -4(\lambda-1) & \cdots & -(2k-2) & -(n-4k) \\ -(2k-2) & \lambda(\lambda-1) & \lambda^2-\lambda+2 & \cdots & \lambda-(2k-2) & -(n-4k) \\ -(2k-2) & -2(\lambda-1) & \lambda^2-\lambda+2 & \cdots & \lambda-(2k-2) & -(n-4k) \\ -(2k-2) & -2(\lambda-1) & -4(\lambda-1) & \cdots & \lambda-(2k-2) & -(n-4k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(2k-2) & -2(\lambda-1) & -4(\lambda-1) & \cdots & \lambda-(2k-2) & -(n-4k) \\ -2k & -2(\lambda-1) & -4(\lambda-1) & \cdots & -2k & \lambda-n+4k+1 \end{vmatrix}.$$

Step 5: In $\det(F)$, replacing R_i by $R_i' = R_i - R_{i+1}$, for $i = 2, 3, \ldots, k, k+1$, it reduces to

$$\det(F) = (\lambda^2 - 3\lambda - 2)^{k-1} \begin{vmatrix} \lambda - 2k + 1 & 2 - 2k & 4k - n \\ 2 & \lambda + 2 & -\lambda - 1 \\ -2k & -2k & \lambda - n + 4k + 1 \end{vmatrix}$$
$$= (\lambda^2 - 3\lambda - 2)^{k-1}(\lambda^3 - (n-4)\lambda^2 - (10k - 3n + 1)\lambda + 2(n - 3k - 1)).$$

Thus, $\phi(A_c(K_{d_n}(k)), \lambda) = (\lambda+1)^{n-4k-1}(\lambda^2 + 3\lambda - 2)^{k-1}(\lambda^2 + \lambda - 4)^k(\lambda^3 - (n-4)\lambda^2 - (3n - 10k - 1)\lambda + 2n - 6k - 2)$. ▷

Example 2.14.

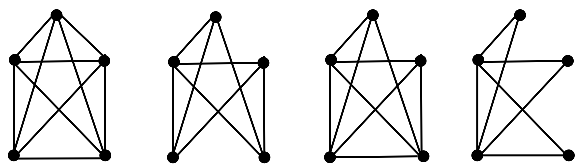

Fig. 1. K_5, $Ka_5(2,2)$, $K_{b_5}(1)$ and $K_{c_5}(3)$.

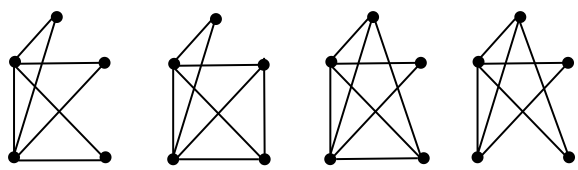

Fig. 2. $K_{f_5}(1)$, $K_{a_5}(2)$, $K_{e_5}(1)$ and $K_{d_5}(1)$.

3. Color Energy of Bipartite Cluster Graphs

Definition 3.1 [12]. Let e_i, $i = 1, 2, \ldots, k$, $1 \leqslant k \leqslant \min\{m, n\}$, be independent edges of the complete bipartite graph $K_{m,n}$, $m, n \geqslant 1$. The cluster graph $Ka_{m,n}(k)$ is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from $K_{m,n}$.

Example 3.2.

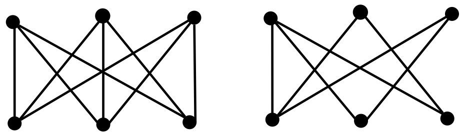

Fig. 3. $K_{3,3}$ and $Ka_{3,3}(2)$.

Definition 3.3 [12]. Let $K_{m,n}$ be the complete bipartite graph, $1 \leqslant r \leqslant m$, $1 \leqslant s \leqslant n$ and $m, n \geqslant 1$. The cluster graph $Kb_{m,n}(r, s)$ is obtained by deleting the edges of $K_{r,s}$ from $K_{m,n}$.

Example 3.4.

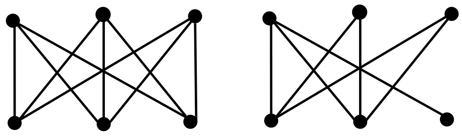

Fig. 4. $K_{3,3}$ and $Kb_{3,3}(2, 1)$.

Theorem 3.5. *For $m, n \geqslant 1$ and $0 \leqslant k \leqslant \min\{m, n\}$, $E_c(Ka_{m,n}(k)) = m + n - 3 + \sqrt{(m + n + 1)^2 - 8k}$.*

$\lhd$ Let $U = \{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be the partites of compete bipartite graph $K_{m,n}$. The cluster graph $Ka_{m,n}(k)$ obtained by deleting independent edges e_i of the complete bipartite graph $K_{m,n}$, $i = 1, 2, \ldots, k$. As $\chi(Ka_{m,n}(k)) = 2$.

We have

$$A_c\left(Ka_{m,n}(k)\right) = \left[\begin{array}{c|c|c|c} (I-J)_k & -J_{k\times(m-k)} & (J-I)_k & J_{k\times(n-k)} \\ \hline -J_{(m-k)\times k} & (I-J)_{(m-k)} & J_{(m-k)\times k} & J_{(m-k)\times(n-k)} \\ \hline (J-I)_k & J_{k\times(m-k)} & (I-J)_k & -J_{k\times(n-k)} \\ \hline J_{(n-k)\times k} & J_{(n-k)\times(m-k)} & -J_{(n-k)\times k} & (I-J)_{(n-k)} \end{array}\right].$$

Consider $\det\left(\lambda I - A_c\left(Ka_{m,n}(k)\right)\right)$.

Step 1: Replace R_i by $R_i' = R_i - R_{i+1}$, for $i = u_1, u_2, \ldots, u_{k-1}, u_{k+1}, u_{k+2}, \ldots, u_{m-1}, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{n-1}$. Then, $\det\left(\lambda I - A_c\left(Ka_{m,n}(k)\right)\right)$ reduces to a new determinant, say

$$\det(C) = \left|\begin{array}{c|c|c|c} X_{k-1\times k} & 0_{k-1\times m-k} & (I-A)_{k-1\times k} & 0_{k-1\times n-k} \\ \hline Y_{1\times k} & J_{1\times m-k} & Z_{1\times k} & -J_{1\times n-k} \\ \hline 0_{m-n-1\times k} & X_{m-k-1\times k} & 0_{m-k-1\times k} & 0_{m-k-1\times n-3k} \\ \hline 1_{1\times k} & Y_{1\times m-k} & -J_{1\times k} & -J_{1\times n-k} \\ \hline Z_{1\times k} & -J_{1\times m-k} & Y_{1\times k} & J_{1\times n-k} \\ \hline 0_{n-k-1\times k} & 0_{n-k-1\times m-k} & 0_{n-k-1\times k} & X_{n-k-1\times n-k} \\ \hline -J_{1\times k} & -J_{1\times m-k} & J_{1\times k} & Y_{1\times n-k} \end{array}\right|,$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$X = (\lambda+1)I - (\lambda+1)B$, $Y = [1\ 1 \ldots 1\ \lambda]$ and $Z = [-1\ \ -1 \ldots -1\ 0]$.

In $\det(C)$, replacing C_i by

$$C_i' = \begin{cases} C_i + C_{i-1} + \ldots + C_1, & \text{for } i = u_k, u_{k-1}, \ldots, u_2; \\ C_i + C_{i-1} + \ldots + C_1, & \text{for } i = v_k, v_{k-1}, \ldots, v_2; \\ C_i + C_{i-1} + \ldots + C_{k+1}, & \text{for } i = u_m, u_{m-1}, \ldots, u_{k+2}; \\ C_i + C_{i-1} + \ldots + C_{k+1}, & \text{for } i = v_n, v_{n-1}, \ldots, v_{k+2}. \end{cases}$$

We obtain

$$\det(C)$$

$$= \left|\begin{array}{c|c|c|c|c|c|c|c} (\lambda-1)I_{k-1} & 0_{k-1\times1} & 0_{k-1\times r} & 0_{k-1\times1} & I_{k-1} & 0_{k-1\times1} & 0_{k-1\times q} & 0_{k-1\times1} \\ \hline M & \lambda-k+1 & N & m-k & -M & -k+1 & -P & -n+k \\ \hline 0_{r\times k-1} & 0_{r\times1} & (\lambda-1)I_r & 0_{m-k\times1} & 0_{r\times k-1} & 0_{r\times1} & 0_{r\times q} & 0_{r\times1} \\ \hline M & k & N & \lambda+r & -M & -k & -P & -n+k \\ \hline I_{k-1} & 0_{k-1\times1} & 0_{k-1\times r} & 0_{k-1\times1} & (\lambda-1)I_{k-1} & 0_{k-1\times1} & 0_{k-1\times q} & 0_{k-1\times1} \\ \hline -M & -k+1 & N & -m+k & M & \lambda+k-1 & P & -n+k \\ \hline 0_{q\times k-1} & 0_{q\times1} & 0_{q\times k-1} & 0_{q\times1} & 0_{q\times k-1} & 0_{q\times1} & (\lambda-1)I_q & 0_{q\times1} \\ \hline -M & -k & -N & -m+k & M & k & P & \lambda+q \end{array}\right|,$$

where $q = n-k-1$, $r = m-k-1$, $M_{1\times k-1} = [1\ 2 \ldots k-1]$, $N_{1\times m-k-1} = [1\ 2\ m-k-1]$ and $P_{1\times n-k-1} = [1\ 2 \ldots n-k-1]$.

Step 3: In $\det(C)$, replacing C_i by $C_i' = (\lambda-1)C_i - C_j$, for $i = v_1, v_2, \ldots, v_{k-1}$ and $j = u_1, u_2, \ldots, u_{k-1}$ and on expanding the $\det(C)$ successively along the rows R_i, for $i = u_1, u_2, \ldots, u_{k-1}, u_{k+1}, u_{k+2}, \ldots, u_{m-1},\ v_1, v_2, \ldots, v_{k-1}, v_{k+1},\ v_{k+2}, \ldots, v_{n-1}$, it simplifies to $(\lambda-1)^{m+n-2k-1}(\lambda^2-2\lambda)^{k-1}\det(D)$, where

$$\det(D) = \begin{vmatrix} \lambda+k+1 & m-k & -(k-1) & -(n-k) \\ k & \lambda+m-k-1 & -k & -(n-k) \\ -(k-1) & -(m-k) & \lambda+k-1 & n-k \\ -k & -(m-k) & k & \lambda+n-k-1 \end{vmatrix}.$$

Now, replacing R_1 by $R_1' = R_1 + R_3$ and replacing R_2 by $R_2' = R_2 + R_4$ it reduces to

$$\det(D) = \begin{vmatrix} \lambda & 0 & \lambda & 0 \\ 0 & \lambda-1 & 0 & \lambda-1 \\ -(k-1) & -(m-k) & \lambda+k-1 & n-k \\ -k & -(m-k) & k & \lambda+n-k-1 \end{vmatrix}.$$

Replacing C_3 by $C_3' = C_3 - C_1$ and replacing C_4 by $C_4' = C_4 - C_2$, we obtain

$$\begin{aligned} \det(D) &= \lambda(\lambda-1)\begin{vmatrix} \lambda+2k-2 & n+m-2k \\ 2k & \lambda+m+n-2k-1 \end{vmatrix} \\ &= \lambda(\lambda-1)(\lambda^2+(m+n-3)\lambda+2(k-m-n+1)). \end{aligned}$$

Thus, $\phi(A_c(Ka_{m,n}(k)),\lambda) = \lambda^k(\lambda-1)^{m+n-2k-1}(\lambda-2)^{k-1}(\lambda^2+(m+n-3)\lambda+2(k-m-n+1))$. So, the color spectrum of $Ka_{m,n}(k)$ is

$$\Big\{0\ (k \text{ times}), 1\ (m+n-2k-1 \text{ times}), 2\ (k-1 \text{ times}), \frac{(3-n-m)+\sqrt{(m+n+1)^2-8k}}{2}, \frac{(3-n-m)-\sqrt{(m+n+1)^2-8k}}{2}\Big\}.$$

Hence, $E_c(Ka_{m,n}(k)) = m+n-3+\sqrt{(m+n+1)^2-8k}$. $\rhd$

Theorem 3.6. *For $m,n \geqslant 1$ and $0 \leqslant r \leqslant m$, $0 \leqslant s \leqslant n$, $\phi(A_c(Kb_{m,n}(r,s)),\lambda) = (\lambda-1)^{m+n-3}(\lambda^3+(m+n-3)\lambda^2+(rs-2n-2m+3)\lambda-(((n+m+1)r-r^2)s-rs^2-n-m+1))$.*

$\lhd$ Let $U = \{u_1,u_2,\dots,u_r,u_{r+1},\dots,u_m\}$ and $V = \{v_1,v_2,\dots,v_s,v_{s+1},\dots,v_n\}$ be the partites of compete bipartite graph $K_{m,n}$. The cluster graph $Kb_{m,n}(r,s)$ obtained by deleting the edges of $K_{r,s}$ from $K_{m,n}$. Since $\chi(Kb_{m,n}(r,s)) = 2$. We have

$$A_c\left(K_{b_{m,n}}(r,s)\right) = \left[\begin{array}{c|c|c|c} (I-J)_r & -J_{r\times(m-r)} & 0_{r\times s} & J_{r\times(n-s)} \\ \hline -J_{m-r\times r} & (I-J)_{(m-r)} & J_{(m-r)\times s} & J_{(m-r)\times(n-s)} \\ \hline 0_{s\times r} & J_{s\times(m-r)} & (I-J)_s & -J_{s\times(n-s)} \\ \hline J_{n-s\times r} & J_{(n-s)\times(m-r)} & -J_{(n-s)\times s} & (I-J)_{(n-s)} \end{array}\right].$$

Consider $\det\left(\lambda I - A_c\left(Kb_{m,n}(r,s)\right)\right)$.

Step 1: Replace R_i by $R_i' = R_i - R_{i+1}$, for $i = 1,2,\dots,r-1,r+1,r+2,\dots,m-1,1,2,\dots,s-1,s+1,s+2,\dots,n-1$. Then, $\det\left(\lambda I - A_c\left(Kb_{m,n}(r,s)\right)\right)$ reduces to $(\lambda-1)^{m+n-4}\det(C)$, where

$$\det(C) = \left|\begin{array}{c|c|c|c} (I-X)_{r-1\times r} & 0_{r-1\times m-r} & 0_{r-1\times s} & 0_{r-1\times n-s} \\ \hline Y_{1\times r} & J_{1\times m-r} & 0_{1\times s} & -J_{1\times n-s} \\ \hline 0_{m-r-1\times r} & (I-X)_{m-r-1\times m-r} & 0_{m-r-1\times s} & 0_{m-r-1\times n-s} \\ \hline J_{1\times r} & Y_{1\times m-r} & -J_{1\times s} & -J_{1\times n-s} \\ \hline 0_{s-1\times r} & 0_{s-1\times m-r} & (I-X)_{s-1\times s} & 0_{s-1\times n-s} \\ \hline 0_{1\times r} & -J_{1\times m-r} & Y_{1\times n-s} & J_{1\times n-s} \\ \hline -J_{1\times r} & -J_{1\times m-r} & J_{1\times n-s} & Z_{1\times n-s} \\ \hline 0_{n-s-1\times r} & 0_{n-s-1\times m-r} & 0_{n-s-1\times n-s} & -(I-X)_{n-s-11\times n-s} \end{array}\right|,$$

where $Y = [1\ 1\dots 1\ \lambda]$, $Z = [\lambda\ 1\dots 1\ 1]$ and

$$X = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Step 2: In $\det(C)$, replacing C_i by

$$C_i' = \begin{cases} C_i + C_{i-1} + \ldots + C_1, & \text{for } i = r, r-1, \dots, 2; \\ C_i + C_{i-1} + \ldots + C_1, & \text{for } i = s, s-1, \dots, 2; \\ C_i + C_{i-1} + \ldots + C_{r+1}, & \text{for } i = m, m-1, \dots, r+2; \\ C_i + C_{i-1} + \ldots + C_{s+1}, & \text{for } i = n, n-1, \dots, s+2, \end{cases}$$

a new determinant $\det(D)$ is obtained.

Step 3: On expanding the $\det(D)$ successively along the rows R_i, for $i = 1, 2, \ldots, r-1, r+1, r+2, \ldots, m-1, 1, 2, \ldots, s-1, s+1, s+2, \ldots, n-1$, it becomes

$$\det(D) = \begin{vmatrix} \lambda + r - 1 & m - r & 0 & -(n-s) \\ r & \lambda + m - r - 1 & -s & -(n-s) \\ 0 & -(m-r) & \lambda + s - 1 & n - s \\ -r & -(m-r) & s & \lambda + n - s - 1 \end{vmatrix}.$$

Replacing, R_4 by $R_4' = R_4 + R_2$, we obtain

$$\det(D) = \begin{vmatrix} \lambda + r - 1 & m - r & 0 & -(n-s) \\ r & \lambda + m - r - 1 & -s & -(n-s) \\ 0 & -(m-r) & \lambda + s - 1 & n - s \\ -r & \lambda - 1 & 0 & \lambda - 1 \end{vmatrix}.$$

Replacing C_2 by $C_2' = C_2 - C_4$, it reduces to

$$\det(D) = (\lambda - 1) \begin{vmatrix} \lambda + r - 1 & m + n - r - s & 0 \\ r & \lambda + m + n - r - s - 1 & -s \\ 0 & r - m - n + s & \lambda + s - 1 \end{vmatrix}$$
$$= (\lambda - 1)(\lambda^3 + (m + n - 3)\lambda^2 + (rs - 2n - 2m + 3)\lambda$$
$$-(((n + m + 1)r - r^2)s - rs^2 - n - m + 1)).$$

Thus, $\phi(A_c(Kb_{m,n}(r,s)), \lambda) = (\lambda - 1)^{m+n-3}(\lambda^3 + (m + n - 3)\lambda^2 + (rs - 2n - 2m + 3)\lambda - (((n + m + 1)r - r^2)s - rs^2 - n - m + 1))$. $\triangleright$

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ЦВЕТОВАЯ ЭНЕРГИЯ НЕКОТОРЫХ КЛАСТЕРНЫХ ГРАФОВ

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Аннотация. Пусть G — простой связный граф. Энергия графа G определяется как сумма абсолютных собственных значений матрицы смежности графа G. Она представляет собой надлежащее обобщение формулы, справедливой для полной энергии π-электронов сопряженного углеводорода, рассчитанной методом молекулярных орбиталей Хюккеля (HMO) в квантовой химии. Раскраской графа G называется раскраска его вершин, при которой никакие две соседние вершины не имеют одинаковый цвет. Минимальное количество цветов, необходимое для раскраски графа G, называется хроматическим числом G и обозначается символом $\chi(G)$. Цветовая энергия графа G определяется как сумма модулей цветовых собственных значений значения G. Графы с большим количеством ребер называют кластерными графами. Кластерный граф — это граф, полученный из полного графа путем удаления несколько ребер в соответствии с некоторыми правилами. Его можно получить, удалив несколько ребер, инцидентных на вершине, удаление независимых ребер/треугольников/клик/пути P3 и т. д. Двудольные кластерные графы получаются удалением нескольких ребер из полного двудольного графа в соответствии с некоторым правилом. В этой статье изучаются цветовая энергия кластерных графов и двудольные кластерные графы.

Ключевые слова: цветовая матрица смежности, цветовое собственное значение, световая энергия.

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