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A NOTE ON SEMIDERIVATIONS IN PRIME RINGS AND $\mathscr{C}^*$-ALGEBRAS[#]

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Abstract. Let $\mathscr{R}$ be a prime ring with the extended centroid $\mathscr{C}$ and the Matrindale quotient ring $\mathscr{Q}$. An additive mapping $\mathscr{F}:\mathscr{R}\to\mathscr{R}$ is called a semiderivation associated with a mapping $\mathscr{G}:\mathscr{R}\to\mathscr{R}$, whenever $\mathscr{F}(xy)=\mathscr{F}(x)\mathscr{G}(y)+x\mathscr{F}(y)=\mathscr{F}(x)y+\mathscr{G}(x)\mathscr{F}(y)$ and $\mathscr{F}(\mathscr{G}(x))=\mathscr{G}(\mathscr{F}(x))$ holds for all $x,y\in\mathscr{R}$. In this manuscript, we investigate and describe the structure of a prime ring $\mathscr{R}$ which satisfies $\mathscr{F}(x^m\circ y^n)\in\mathscr{Z}(\mathscr{R})$ for all $x,y\in\mathscr{R}$, where $m,n\in\mathbb{Z}^+$ and $\mathscr{F}:\mathscr{R}\to\mathscr{R}$ is a semiderivation with an automorphism ξ of $\mathscr{R}$. Further, as an application of our ring theoretic results, we discussed the nature of $\mathscr{C}^*$-algebras. To be more specific, we obtain for any primitive $\mathscr{C}^*$-algebra $\mathscr{A}$. If an anti-automorphism $\zeta:\mathscr{A}\to\mathscr{A}$ satisfies the relation $(x^n)^\zeta+x^{n*}\in\mathscr{Z}(\mathscr{A})$ for every $x,y\in\mathscr{A}$, then $\mathscr{A}$ is $\mathscr{C}^*-\mathscr{W}_4$-algebra, i. e., $\mathscr{A}$ satisfies the standard identity $\mathscr{W}_4(a_1,a_2,a_3,a_4)=0$ for all $a_1,a_2,a_3,a_4\in\mathscr{A}$.

Key words: prime ring, automorphism, semiderivation.

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1. Introduction

Throughout the paper unless otherwise stated, $\mathscr{R}$ is the prime ring with centre $\mathscr{Z}(\mathscr{R})$, $\mathscr{Q}$ is the Martindale quotient ring of $\mathscr{R}$ and $\mathscr{C}$ is the extended centroid $\mathscr{R}$ (for further details see [1]). For given $x,y\in\mathscr{R}$, the symbol $[x,y]$ and $x\circ y$ stands for the commutator and anti-commutator of x and y defined as $xy-yx$ and $xy+yx$, respectively. We also note that a ring $\mathscr{R}$ is said to be a prime ring if $a\mathscr{R}b=\{0\}$ implies that either $a=0$ or $b=0$. For any subsets $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{R}$, $[\mathscr{A},\mathscr{B}]$ stands for the additive subgroup generated by $[a,b]$ with $a\in\mathscr{A}$ and $b\in\mathscr{B}$. Also, an additive subgroup $\mathscr{L}$ of $\mathscr{R}$ is said to be Lie ideal of $\mathscr{R}$ if $[u,r]\in\mathscr{L}$ for all $u\in\mathscr{L}$ and $r\in\mathscr{R}$. A mapping $g:\mathscr{R}\to\mathscr{R}$ is said to be commuting (resp. centralizing) on a subset $\mathscr{S}$ of $\mathscr{R}$ if $[g(x),x]=0$ (resp. $[g(x),x]\in\mathscr{Z}(\mathscr{R})$) for all $x\in\mathscr{S}$. An additive mapping $\mathscr{D}:\mathscr{R}\to\mathscr{R}$ is called a derivation on $\mathscr{R}$, if $\mathscr{D}(xy)=\mathscr{D}(x)y+x\mathscr{D}(y)$ holds for all $x,y\in\mathscr{R}$.

In [2], Bergen introduced the notion of semiderivation. An additive mapping $\mathscr{F}:\mathscr{R}\to\mathscr{R}$ is called a semiderivation associated with a mapping $\mathscr{G}:\mathscr{R}\to\mathscr{R}$, whenever

$$\mathscr{F}(xy)=\mathscr{F}(x)\mathscr{G}(y)+x\mathscr{F}(y)=\mathscr{F}(x)y+\mathscr{G}(x)\mathscr{F}(y)$$

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and $\mathscr{F}(\mathscr{G}(x)) = \mathscr{G}(\mathscr{F}(x))$ holds for all $x, y \in \mathscr{R}$. For $\mathscr{G} = 1_{\mathscr{R}}$, the identity map on $\mathscr{R}$, $\mathscr{F}$ is clearly a derivation. Brešer [3] proved that the only semiderivations of prime rings are ordinary derivations and mappings of the form $\mathscr{F}(x) = \gamma(x - \mathscr{G}(x))$, where $\gamma \in \mathscr{C}$ and $\mathscr{G}$ is an endomorphism.

Let us briefly recall the motivation behind this study. In [4], Posner studied the centralizing derivations of prime rings and proved that if $\mathscr{R}$ is a prime ring and $\mathscr{D}$ is a non-zero derivation of $\mathscr{R}$ such that $[\mathscr{D}(x), x] \in \mathscr{Z}(\mathscr{R})$, for all $x \in \mathscr{R}$, then $\mathscr{R}$ is commutative. This result due to Posner was then extended to Lie ideals by Lanski [5]. In [6], Daif and Bell showed that a semiprime ring $\mathscr{R}$ must be commutative if it admits a derivation $\mathscr{D}$ such that either $\mathscr{D}([x, y]) - [x, y] = 0$ for all $x, y \in \mathscr{R}$ or $\mathscr{D}([x, y]) + [x, y] = 0$ for all $x, y \in \mathscr{R}$. In 2002, Ashraf and Rehman [7] obtained the same conclusion if the commutator is replaced by an anti-commutator which stated that if a prime ring $\mathscr{R}$ admits a derivation $\mathscr{D}$ such that $\mathscr{D}(x) \circ \mathscr{D}(y) = x \circ y$ for all $x, y \in \mathscr{R}$, then $\mathscr{R}$ is commutative. In [8], Herstein proved that a ring $\mathscr{R}$ is commutative if it has no nonzero nilpotent ideal and there is a fixed integer $n > 1$ such that $(xy)^n = x^n y^n$ for all $x, y \in \mathscr{R}$. In [9], Bell proved that a prime ring $\mathscr{R}$ with nonzero center, for which $\text{char}(\mathscr{R}) = 0$ or $\text{char}(\mathscr{R}) > n$, where $n > 1$, must be commutative if it admits a nonzero derivation $\mathscr{D}$ such that $\mathscr{D}([x^n, y] - [x, y^n]) \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{R}$. Further, Ali et al. [10] showed that if $\mathscr{R}$ be a 2-torsion free semiprime ring and it admits a derivation $\mathscr{D}$ such that $\mathscr{D}(x^m \circ y^n) \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{R}$, then $\mathscr{R}$ is commutative (for additional associated results [11–14]).

On the other hand, recently Haung [15] proved that a prime ring $\mathscr{R}$ satisfies s_4, the standard identity in four variables if $\text{char}(\mathscr{R}) > n + 1$ or $\text{char}(\mathscr{R}) = 0$ and $\mathscr{F}(x)^n = 0$ holds, where $x \in \mathscr{L}$, a noncentral Lie ideal of $\mathscr{R}$ and $\mathscr{F}$ is a semiderivation associated with an automorphism ξ of $\mathscr{R}$.

Given the above discussions, we investigate and describe the structure of a ring $\mathscr{R}$ which satisfies certain identities involving automorphisms and semi-derivations. Also, we discuss the nature of $\mathscr{C}^*$-algebras. To be more specific, we obtain the following theorems:

Theorem 1.1. *Let $\mathscr{R}$ be a prime ring of* $\text{char}(\mathscr{R}) \neq 2$ *and $m, n \in \mathbb{Z}^+$. If an automorphism ζ of $\mathscr{R}$ satisfies $(x^m \circ y^n)^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

Theorem 1.2. *Let $\mathscr{R}$ be a prime ring of* $\text{char}(\mathscr{R}) \neq 2$ *and $m, n \in \mathbb{Z}^+$. If a semiderivation $\mathscr{F}$ associated with an automorphism ξ such that $\mathscr{F}(x^m \circ y^n) \in \mathscr{Z}(\mathscr{R})$. Then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

Theorem 1.3. *Let $\mathscr{A}$ be a primitive $\mathscr{C}^*$-algebra and $m, n \in \mathbb{Z}^+$. If an automorphism $\xi : \mathscr{A} \to \mathscr{A}$ satisfies the relation $(x^m \circ y^n)^\zeta \in \mathscr{Z}(\mathscr{A})$ for all $x, y \in \mathscr{A}$, then $\mathscr{A}$ is $\mathscr{C}^* - \mathscr{W}_4$-algebra.*

Theorem 1.4. *Let $\mathscr{A}$ be a primitive $\mathscr{C}^*$-algebra and $n \in \mathbb{Z}^+$. If an anti-automorphism $\zeta : \mathscr{A} \to \mathscr{A}$ satisfies the relation $(x^n)^\zeta + x^{n*} \in \mathscr{Z}(\mathscr{A})$ for every $x, y \in \mathscr{A}$, then $\mathscr{A}$ is $\mathscr{C}^* - \mathscr{W}_4$-algebra.*

2. Preliminaries

Before proving our main results, we fix some notions which are required for the exposition of our main results. An automorphism ξ is called $\mathscr{Q}$-inner if there exists an invertible element $q \in \mathscr{Q}$ such that $\xi(x) = qxq^{-1}$ for all $x \in \mathscr{R}$. Also, the standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^\mu \mathtt{X}_{\mu(1)} \mathtt{X}_{\mu(2)} \mathtt{X}_{\mu(3)} \mathtt{X}_{\mu(4)},$$

where $(-1)^{\mu}$ is a sign of permutation μ of the symmetric group of degree 4. Further we mention the following results which are crucial in developing the proof of our main theorem.

Fact 2.1. *Let $\mathscr{R}$ be a prime ring and $\mathscr{I}$ a two sided ideal of $\mathscr{R}$. Then $\mathscr{I}$, $\mathscr{R}$, $\mathscr{Q}$ satisfy the same generalized polynomial identities with coefficients in $\mathscr{Q}$* (see [16]). *Furthermore, $\mathscr{I}$, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identities with automorphisms* (see [17, Theorem 1]).

Fact 2.2. *Let $\mathscr{R}$ be a prime ring with extended centroid $\mathscr{C}$. Then the following conditions are equivalent:*

(i) $\dim_{\mathscr{C}} \mathscr{R}\mathscr{C} \leqslant 4$.

(ii) *$\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

(iii) *$\mathscr{R}$ is commutative or R embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field.*

(iv) *$\mathscr{R}$ is algebraic of bounded degree 2 over $\mathscr{C}$.*

(v) *$\mathscr{R}$ satisfies $[[x^2, y], [x, y]] = 0$.*

Fact 2.3. *Let $\mathscr{R}$ be a prime ring and $\mathscr{L}$ a be non-central Lie ideal of $\mathscr{R}$. If $\mathrm{char}(\mathscr{R}) \neq 2$, by* [18, Lemma 1] *there exists a nonzero ideal $\mathscr{I}$ of $\mathscr{R}$ such that $0 \neq [\mathscr{I}, \mathscr{R}] \subseteq \mathscr{L}$. If $\mathrm{char}(\mathscr{R}) = 2$ and $\dim_{\mathscr{C}} \mathscr{R}\mathscr{C} > 4$, i. e., $\mathrm{char}(\mathscr{R}) = 2$ and $\mathscr{R}$ does not satisfy s_4, then by* [19, Theorem 13] *there exists a nonzero ideal $\mathscr{I}$ of $\mathscr{R}$ such that $0 \neq [\mathscr{I}, \mathscr{R}] \subseteq \mathscr{L}$. Thus if either $\mathrm{char}(\mathscr{R}) \neq 2$ or $\mathscr{R}$ does not satisfy s_4, then we may conclude that there exists a nonzero ideal $\mathscr{I}$ of $\mathscr{R}$ such that $[\mathscr{I}, \mathscr{I}] \subseteq \mathscr{L}$.*

3. Main Results

Proposition 3.1. *Let $\mathscr{R}$ be a dense subring of $End(\mathscr{V}_{\mathscr{D}})$ and $\zeta : \mathscr{R} \to \mathscr{R}$ be an automorphism of $\mathscr{R}$. If $\mathscr{R}$ satisfies $([x_1, x_2] \circ [y_1, y_2])^{\zeta} \in \mathscr{Z}(\mathscr{R})$ for all $x_1, x_2, y_1, y_2 \in \mathscr{R}$, then either $\dim(\mathscr{V}_{\mathscr{D}}) \leqslant 2$ or ζ is an identity map on $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$.*

◁ First assume that $\mathscr{V}_{\mathscr{D}}$ be a right vector space over a division ring $\mathscr{D}$. Let $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$ the ring of $\mathscr{D}$-linear transformations on $\mathscr{V}_{\mathscr{D}}$. Thus in view of classical Jacobson Theorem [20, Isomorphism Theorem, p. 79], we have $s^{\zeta} = \mathscr{P} s \mathscr{P}^{-1}$ for every $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$, where ζ is an automorphism of $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$ and $\mathscr{P}$ is an invertible semi-linear transformation. Hence, for all $v \in \mathscr{V}$, $\zeta \in \mathscr{D}$, $\mathscr{P}(v\varphi) = (\mathscr{P}v)\zeta(\varphi)$. Given by the hypotheses, we obtain

$$0 = \big[[x_1, x_2]^{\zeta}[y_1, y_2]^{\zeta} + [y_1, y_2]^{\zeta}[x_1, x_2]^{\zeta}, z\big] = \big[\mathscr{P}[x_1, x_2][y_1, y_2]\mathscr{P}^{-1} + \mathscr{P}[y_1, y_2][x_1, x_2]\mathscr{P}^{-1}, z\big]$$

for every $x_1, x_2, y_1, y_2, z \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. Let us assume that v and $\mathscr{P}^{-1}v$ are $\mathscr{D}$-dependent for every $v \in \mathscr{V}$. In view of [21, Lemma 1], we find that $\mathscr{P}^{-1}v = v\chi$, where $\chi \in \mathscr{D}$ and $v \in \mathscr{V}$. Hence, for all $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$, $\mathscr{P}^{-1}(sv) = sv\chi$ and $sv = \mathscr{P}(sv\chi) = \mathscr{P}(s(v\chi)) = \mathscr{P}s\mathscr{P}^{-1}(v) = s^{\zeta}v$ for all $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$, $v \in \mathscr{V}$. Therefore, we find that $(s^{\zeta} - s)\mathscr{V} = (0)$ for every $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. Hence, $s^{\zeta} = s$ for every $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. This shows that ζ is an identity map on $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$, as required.

Thus, there exists $v \in \mathscr{V}$ such that v and $\mathscr{P}^{-1}v$ are linearly $\mathscr{D}$-independent. Firstly, we assume that $\dim(\mathscr{V}_{\mathscr{D}}) \geqslant 4$. Then we may take $w, \mathscr{P}v \in \mathscr{V}$ such that $\{w, v, \mathscr{P}v, \mathscr{P}^{-1}v\}$ is $\mathscr{D}$-independent. Let $x, y \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$ such that

$$x_1 v = 0, \quad x_1\mathscr{P}^{-1}v = 0, \quad x_1 w = v, \quad y_1\mathscr{P}^{-1}v = 0, \quad zv = 0;$$
$$x_2 v = w, \quad x_2\mathscr{P}^{-1}v = v, \quad y_1 v = v, \quad y_2\mathscr{P}^{-1}v = v, \quad z\mathscr{P}v = w.$$

We notice that $[x_1, x_2]\mathscr{P}^{-1}v = 0$, $[y_1, y_2]\mathscr{P}^{-1}v = v$, $[x_1, x_2]v = v$ and hence, our assumption yields

$$0 = \Big(\Big[\mathscr{P}[x_1, x_2][y_1, y_2]\mathscr{P}^{-1} + \mathscr{P}[y_1, y_2][x_1, x_2]\mathscr{P}^{-1}, z\Big]\Big)v = -w,$$

a contradiction, implying that $\dim(\mathscr{V}_{\mathscr{D}}) \leqslant 3$.

Secondly, we assume that $\dim(\mathscr{V}_{\mathscr{D}}) = 3$. Take $\mathscr{P}v \in \mathscr{V}$ such that $\{v, \mathscr{P}v, \mathscr{P}^{-1}v\}$ is $\mathscr{D}$-independent and then $\{v, \mathscr{P}v, \mathscr{P}^{-1}v\}$ forms a $\mathscr{D}$-basis of $\mathscr{V}$. If $\mathscr{P}(v + \mathscr{P}^{-1}v + \mathscr{P}v) \in v\mathscr{D}$ and $\mathscr{P}(\mathscr{P}^{-1}v + \mathscr{P}v) \in v\mathscr{D}$, then $\mathscr{P}v, \mathscr{P}(\mathscr{P}^{-1}v + \mathscr{P}v) \in v\mathscr{D}$ and then $v, \mathscr{P}^{-1}v + \mathscr{P}v \in \mathscr{P}^{-1}(v\mathscr{D}) = \mathscr{P}^{-1}(v)\zeta^{-1}(\mathscr{D}) = \mathscr{P}^{-1}v\mathscr{D}$, contradicting the fact that $\{v, \mathscr{P}^{-1}v + \mathscr{P}v\}$ is $\mathscr{D}$-independent. Therefore, one can pick $\rho \in \{0, 1\}$ such that $u = \rho v + \mathscr{P}^{-1}v + \mathscr{P}v$ and $\mathscr{P}u \notin v\mathscr{D}$. Write $\mathscr{P}u = v\alpha + \mathscr{P}^{-1}v\beta + \mathscr{P}v\gamma$, where $\alpha, \beta, \gamma \in \mathscr{D}$ and β, γ both are not zero. By density of theorem, there exist $x_1, x_2, y_1, y_2, z \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$ such that

$$
\begin{gathered}
x_1 v = 0, \quad x_2 v = \mathscr{P}v, \quad y_1 v = v, \quad y_2 v = v, \quad zv = 0; \\
x_1\mathscr{P}^{-1}v = v, \quad x_2\mathscr{P}^{-1}v = 0, \quad y_1\mathscr{P}^{-1}v = 0, \quad y_2\mathscr{P}^{-1}v = v, \quad z\mathscr{P}^{-1}v = v; \\
x_1\mathscr{P}v = u, \quad x_2\mathscr{P}v = 0, \quad y_1\mathscr{P}v = v, \quad y_2\mathscr{P}v = v, \quad z\mathscr{P}v = u.
\end{gathered}
$$

That is $x_1u = (\rho+1)v + \mathscr{P}^{-1}v + \mathscr{P}v$, $x_2u = \mathscr{P}v, y_1u = (\rho+1)v$ and $y_2u = -u\gamma$. Therefore, we can see that $[x_1, x_2]\mathscr{P}^{-1}v = -\mathscr{P}^{-1}v$, $[y_1, y_2]\mathscr{P}^{-1}v = v$, $[x_1, x_2]v = u$, $[y_1, y_2]\mathscr{P}^{-1}v = 0$. Also, $z\mathscr{P}u = v\beta + u\gamma$. As β, γ are not both zero and v, u are $\mathscr{D}$-dependent, so it is easy to see that $z\mathscr{P}u \neq 0$. Thus in all, we see that

$$
0 = \Big(\Big[\mathscr{P}[x_1, x_2][y_1, y_2]\mathscr{P}^{-1} + \mathscr{P}[y_1, y_2][x_1, x_2]\mathscr{P}^{-1}, z\Big]\Big)v = -z\mathscr{P}u,
$$

a contradiction, implying that $\dim(\mathscr{V}_{\mathscr{D}}) \leqslant 2$. $\rhd$

Theorem 3.1. *Let $\mathscr{R}$ be a non-commutative prime ring of characteristic different from two and ζ be an automorphism of $\mathscr{R}$. If $\mathscr{R}$ satisfies $([x_1, x_2] \circ [y_1, y_2])^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x_1, x_2, y_1, y_2 \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

$\lhd$ Firstly, we assume that ζ is an inner automorphism of $\mathscr{R}$, i. e., $s^\zeta = psp^{-1}$ for every $s \in \mathscr{R}$. As ζ is the non-identity map, so $p \notin \mathscr{C}$. Then

$$
\Psi(r) = \Big[\mathscr{P}[x_1, x_2][y_1, y_2]\mathscr{P}^{-1} + \mathscr{P}[y_1, y_2][x_1, x_2]\mathscr{P}^{-1}, z\Big]
$$

is a non-trivial generalized polynomial identity (GPI) of $\mathscr{R}$ and hence of $\mathscr{Q}$ as well. By Martindale's theorem [22], $\mathscr{Q}$ is isomorphic to dense subring of the ring of linear transformations of a vector space $\mathscr{V}$ over $\mathscr{D}$, where $\mathscr{D}$ is a finite dimensional division ring over $\mathscr{C}$. By Proposition 3.1, we have $\dim(\mathscr{V}_{\mathscr{D}}) \leqslant 2$. Thus it follows that either $\mathscr{Q} \cong \mathscr{D}$ or $\mathscr{Q} \cong \mathscr{M}_2(\mathscr{D})$, the ring of 2×2 matrices over $\mathscr{D}$. More generally, we assume that $\mathscr{Q} \cong \mathscr{M}_k(\mathscr{D})$, for $k \leqslant 2$.

If $\mathscr{C}$ is finite, then $\mathscr{D}$ is field by Wedderburn's theorem. On the other hand, if $\mathscr{C}$ infinite, let $\mathscr{F}$ be the algebraic closure of $\mathscr{C}$, therefore by the Van der monde determinant argument, we see that $\mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F}$ satisfies the generalized polynomial identity $\Psi(r) = 0$. Moreover, $\mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F} \cong \mathscr{M}_k(\mathscr{D}) \otimes_{\mathscr{C}} \mathscr{F} \cong \mathscr{M}_k(\mathscr{D} \otimes_{\mathscr{C}} \mathscr{F}) \cong \mathscr{M}_t(\mathscr{F})$, for some $t \geqslant 1$. Considering Proposition 3.1 and the fact that $\mathscr{Q}$ is not commutative, we assert that $t = 2$, yields the required conclusion.

Secondly, we assume that ζ is an outer automorphism. By [17, Theorem 1], $\mathscr{Q}$ and hence $\mathscr{R}$ satisfy $[[x_1, x_2]^\zeta[y_1, y_2]^\zeta + [y_1, y_2]^\zeta[x_1, x_2]^\zeta, z] = 0$. As x^ζ, y^ζ-word degree $<$ $\mathrm{char}(\mathscr{R})$, then by [23, Theorem 3], $\mathscr{R}$ satisfies $[[x'_1, x'_2][y'_1, y'_2] + [y'_1, y'_2][x'_1, x'_2], z] = 0$. That is, $\mathscr{R}$ is a polynomial identity (PI) ring. Thus, $\mathscr{R}$ and $\mathscr{M}_t(\mathscr{F})$ satisfy the same polynomial identities [24, Lemma 1], i. e., for each $x'_1, x'_2, y'_1, y'_2, z \in \mathscr{M}_t(\mathscr{F})$, $[[x'_1, x'_2][y'_1, y'_2] + [y'_1, y'_2][x'_1, x'_2], z] = 0$. Take $k \geqslant 3$ and e_{ij}, the usual unit matrix. Therefore, for $x = e_{23}$, $y = e_{32}$, $z = e_{11}$, $s = e_{12}$, we get a contradiction $0 = [[x'_1, x'_2][y'_1, y'_2] + [y'_1, y'_2][x'_1, x'_2], z] = [[e_{11}, e_{12}][e_{23}, e_{32}] + [e_{23}, e_{32}][e_{11}, e_{12}], [e_{23}, e_{32}]] = e_{12} \neq 0$. Hence $t = 2$, i. e., $\mathscr{R}$ satisfies s_4, the standard identity in four variables. This completes the proof. $\rhd$

◁ Proof of Theorem 1.1. We are given that $(x^m \circ y^n)^\zeta \in \mathscr{Z}(\mathscr{R})$ for every $x, y \in \mathscr{R}$. Let $S_1 = \{r^m : r \in \mathscr{R}\}$ and $S_2 = \{r^n : r \in \mathscr{R}\}$ be the additive subgroups. It implies that $(a \circ b)^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $a \in S_1$, $b \in S_2$. In view of [25, Main theorem], and since $\mathrm{char}(\mathscr{R}) \neq 2$, either S_1 have a non-central Lie ideal $\mathscr{L}_1$ of $\mathscr{R}$ or $r^m \in \mathscr{Z}(\mathscr{R})$ for all $r \in \mathscr{R}$. The latter case concludes $\mathscr{R}$ to be commutative. Similarly, assume that there exists a Lie ideal $\mathscr{L}_2 \nsubseteq \mathscr{Z}(\mathscr{R})$ such that $\mathscr{L}_2 \subseteq S_2$. Moreover, in view of Fact 2.3, there exist $\mathscr{I}_1$ and $\mathscr{I}_2$ nonzero two-sided ideals of $\mathscr{R}$ such that $0 \neq [\mathscr{I}_1, \mathscr{R}] \subseteq \mathscr{L}_1$ and $0 \neq [\mathscr{I}_2, \mathscr{R}] \subseteq \mathscr{L}_2$. Also, $\mathscr{R}$ is non-commutative as $\mathscr{L}_1, \mathscr{L}_2$ are non-central Lie ideal of $\mathscr{R}$. Therefore $(x \circ y)^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x \in [\mathscr{I}_1, \mathscr{I}_1]$, $y \in [\mathscr{I}_2, \mathscr{I}_2]$. Since $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{R}$ satisfy the same differential identities (see [24, Theorem 3]), so we have $(x \circ y)^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in [\mathscr{R}, \mathscr{R}]$. By Theorem 3.1, we get the required result. ▷

Using the same technique as used in Theorem 1.1 and Theorem 3.1, we can write in view of above result

Theorem 3.2. *Let $\mathscr{R}$ be a non-commutative prime ring of characteristic different from two and $\mathscr{F}$ be a non-zero semiderivation associated with an automorphism ξ of $\mathscr{R}$. If $\mathscr{R}$ satisfies $\mathscr{F}([x_1, x_2] \circ [y_1, y_2]) \in \mathscr{Z}(\mathscr{R})$ for all $x_1, x_2, y_1, y_2 \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

◁ First we note that if ξ is an identity map on $\mathscr{R}$, then $\mathscr{F}$ is not more than a derivation. In view of previous discussion, we have nothing to prove. Hence, we proceed by assuming that ξ is not an identity map on $\mathscr{R}$. Hence in view of Brešar [3], $\mathscr{F}(x) = \gamma(x - x^\xi)$ for all $x \in \mathscr{R}$, where $0 \neq \gamma \in \mathscr{C}$. Thus by our hypothesis we can write $\gamma([x_1, x_2] \circ [y_1, y_2] - ([x_1, x_2] \circ [y_1, y_2])^\xi) \in \mathscr{Z}(\mathscr{R})$ which can be rewritten as $\gamma(([x_1, x_2] \circ [y_1, y_2])^{I_\mathscr{R}} - ([x_1, x_2] \circ [y_1, y_2])^\xi) \in \mathscr{Z}(\mathscr{R})$, where $I_\mathscr{R}$ is the identity map on $\mathscr{R}$. It is well known that if ξ is an automorphism of $\mathscr{R}$, then $\xi + kI_\mathscr{R}$ (k is an any integer) is also an automorphism on $\mathscr{R}$. Thus, we set $\xi - I_\mathscr{R} = \zeta$. Therefore, the last relation can be written as $\gamma([x_1, x_2] \circ [y_1, y_2])^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x_1, x_2, y_1, y_2 \in \mathscr{R}$. Since $0 \neq \gamma \in \mathscr{C}$, the above identity reduces to $([x_1, x_2] \circ [y_1, y_2])^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x_1, x_2, y_1, y_2 \in \mathscr{R}$ and hence in view of Theorem 3.1, we get the desired conclusion. ▷

Proof of Theorem 1.2. We are given that $\mathscr{F}(x^m \circ y^n) \in \mathscr{Z}(\mathscr{R})$ for every $x, y \in \mathscr{R}$. Let $S_1 = \{r^m : r \in \mathscr{R}\}$ and $S_2 = \{r^n : r \in \mathscr{R}\}$ be the additive subgroups. It is easy to see that $\mathscr{F}(x \circ y) \in \mathscr{Z}(\mathscr{R})$ for each $x \in S_1$, $y \in S_2$. Since $\mathrm{char}(\mathscr{R}) \neq 2$ and by main theorem of [25], we have either $r^m \in \mathscr{Z}(\mathscr{R})$ for every $r \in \mathscr{R}$ or S_1 contains a non-central Lie ideal $\mathscr{L}_1$ of $\mathscr{R}$. The first case concludes that $\mathscr{R}$ to be commutative. Similarly, assume that there exists a Lie ideal $\mathscr{L}_2 \nsubseteq Z(\mathscr{R})$ such that $\mathscr{L}_2 \subseteq S_2$. According to Fact 2.3, there exist nonzero two-sided ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ of $\mathscr{R}$ such that $0 \neq [\mathscr{I}_1, \mathscr{R}] \subseteq \mathscr{L}_1$ and $0 \neq [\mathscr{I}_2, \mathscr{R}] \subseteq \mathscr{L}_2$. Since $\mathscr{L}_1, \mathscr{L}_2$ are non-central Lie ideal of $\mathscr{R}$, so $\mathscr{R}$ is non-commutative. Hence, $\mathscr{F}(x \circ y) \in Z(\mathscr{R})$ for all $x \in [\mathscr{I}_1, \mathscr{I}_1], y \in [\mathscr{I}_2, \mathscr{I}_2]$. Since $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{R}$ satisfy the same differential identities (see [24, Theorem 3]), so we have $\mathscr{F}(x \circ y) \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in [\mathscr{R}, \mathscr{R}]$. Applying Theorem 3.2, we are done.

Corollary 3.1. *Let $\mathscr{R}$ be a prime ring of characteristic different from two, m be fixed positive integer and $\mathscr{F}$ be a nonzero semiderivation associated with an automorphism ξ of $\mathscr{R}$. If $\mathscr{F}(x^m) \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

Corollary 3.2. *Let $\mathscr{R}$ be a prime ring of characteristic not two. If $\mathscr{R}$ admits an automorphism ζ of $\mathscr{R}$ such that $(x^n)^\zeta \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

Theorem 3.3. *Let $\mathscr{R}$ be a prime ring of characteristic not two. If $\mathscr{R}$ admits an automorphism ζ of $\mathscr{R}$ such that $(x^n)^\zeta + x^n \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

$\lhd$ It is well known that if ζ is an automorphism of $\mathscr{R}$, then $\zeta + kI_{\mathscr{R}}$ (k is an any integer) is also an automorphism on $\mathscr{R}$. We have given that $(x^n)^\zeta + x^n \in Z(\mathscr{R})$ for all $x \in \mathscr{R}$ which can be rewritten as $(x^n)^\zeta + (x^n)^{I_{\mathscr{R}}} \in Z(\mathscr{R})$, where $I_{\mathscr{R}}$ is the identity map on $\mathscr{R}$. Thus, we set $\zeta - I_{\mathscr{R}} = \xi$. Therefore, the last relation can be written as $(x^n)^\xi \in Z(\mathscr{R})$ for all $x \in \mathscr{R}$ and hence by Corollary 3.2 we have done. $\rhd$

4. Result Based on $\mathscr{C}^*$-Algebras

A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $\|\cdot\|$ satisfying the multiplicative inequality; $\|xy\| \leqslant \|x\|\|y\|$ for all x and y in $\mathscr{A}$. A Banach algebra $\mathscr{A}$ is a PI-algebra if and only if there exists $n \in \mathbb{N}$ and a polynomial $q \in \mathscr{W}_n$, $q \neq 0$, such that $q(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$, where $\mathscr{W}_n$ is the set of all complex polynomials in n non-commuting variables. An involution on an algebra $\mathscr{A}$ is a map $x \longmapsto x^*$ of $\mathscr{A}$ onto such that the following conditions are hold: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(x + \lambda y)^* = x^* + \overline{\lambda} y^*$ for all $x, y \in \mathscr{A}$ and $\lambda \in \mathbb{C}$ the field of complex number, where $\overline{\lambda}$ is the conjugate of λ. Of course the prototypical example of an involution on a Banach algebra is the adjoint operation on $\mathscr{B}(\mathscr{H})$, the set of bounded linear operators on Hilbert space $\mathscr{H}$. Another important example is complex conjugation on $\mathbb{C}(\mathbb{X})$, the set of all continuous complex valued functions on $\mathbb{X}$, a compact Hausdroff space defined as $f^*(x) := \overline{f(x)}$.

An algebra equipped with an involution is called a $*$-algebra or algebra with involution. A Banach $*$-algebra is a Banach algebra $\mathscr{A}$ together with an isometric involution $\|x^*\| = \|x\|$ for all $x \in \mathscr{A}$. A Banach $*$-algebra is called a $\mathscr{C}^*$-algebra $\mathscr{A}$ if $\|x^*x\| = \|x\|^2$ for all $x \in \mathscr{A}$. A $\mathscr{C}^*$-algebra $\mathscr{A}$ is primitive if its zero ideal is primitive, that is, if $\mathscr{A}$ has a faithful non-zero irreducible representation. Let $\mathscr{W}_n$ denote the standard polynomial of degree n in n non-commuting variables, $\mathscr{W}_n = \Sigma_{\sigma \in S_n}$ sign $(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}$, where S_n is the set of all permutations of $\{1, 2, 3, \cdots, n\}$ and $\operatorname{sign}(\sigma) = \pm 1$ for σ even (odd) (see [26, 27] and references therein). An algebra $\mathscr{A}$ is said to be an $\mathscr{C}^*$-$\mathscr{W}_n$-algebra if $\mathscr{W}_n(a_1, a_2, \cdots, a_n) = 0$ for each choice of elements $a_1, a_2, \cdots, a_n \in \mathscr{A}$. In particular, an algebra is $\mathscr{C}^* - \mathscr{W}_4$-algebra if it satisfies the standard identity $\mathscr{W}_4(a_1, a_2, a_3, a_4) = 0$ for all $a_1, a_2, a_3, a_4 \in \mathscr{A}$. Moreover, an algebra is $\mathscr{C}^* - \mathscr{W}_2$-algebra if and only if it is commutative, i. e., a $\mathscr{C}^* - \mathscr{W}_2$-algebra is commutative if it satisfies the standard identity $\mathscr{W}_2(a_1, a_2) = 0$ for all $a_1, a_2 \in \mathscr{A}$. Many researcher discussed Gelfand's theory for Banach algebra and $\mathscr{C}^*$-algebra namely, Banach-$\mathscr{W}_{2n}$-algebra and $\mathscr{C}^* - \mathscr{W}_{2n}$-algebra. Throughout the present section, $\mathscr{C}^*$-algebras are assumed to be nonunital unless indicated otherwise.

$\lhd$ Proof of Theorem 1.3. We have given that $\zeta : \mathscr{A} \to \mathscr{A}$ is an automorphism of $\mathscr{A}$ and $\mathscr{A}$ is a primitive $\mathscr{C}^*$-algebra such that $(x^m \circ y^n)^\zeta \in Z(\mathscr{A})$ for all $x, y \in \mathscr{A}$. Therefore, $\mathscr{A}$ is prime by [28, Theorem 5.4.5] because $\mathscr{A}$ is primitive $\mathscr{C}^*$-algebra. Hence, $\mathscr{A}$ is a prime ring since $\mathscr{A}$ is a prime $\mathscr{C}^*$-algebra. By application of Theorem 1.1 get the required conclusion, thereby proving the theorem. $\rhd$

$\lhd$ Proof of Theorem 1.4. We have $(x^n)^\zeta + x^{n*} \in Z(\mathscr{A})$ for all $x \in \mathscr{A}$. Replace x^* for x, to get $(x^{n*})^\zeta + x^n \in Z(\mathscr{A})$ for all $x \in \mathscr{A}$. Now, a map $\pi : \mathscr{A} \to \mathscr{A}$ by $x^\pi = x^{*\xi}$ for every $x \in \mathscr{A}$. It is easy to see that $(xy)^\pi = x^\pi y^\pi$ for all $x, y \in \mathscr{A}$, that is, π is an automorphism of $\mathscr{A}$ and hence we find that $(x^n)^\pi + x^n \in Z(\mathscr{A})$ for every $x \in \mathscr{A}$. Therefore, $\mathscr{A}$ is prime by [28, Theorem 5.4.5] because $\mathscr{A}$ primitive $\mathscr{C}^*$-algebra. Hence, $\mathscr{A}$ is a prime ring since $\mathscr{A}$ is a prime $\mathscr{C}^*$-algebra. Application of Theorem 3.3 yields the required conclusion. $\rhd$

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ПОЛУДИФФЕРЕНЦИРОВАНИЯ В ПЕРВИЧНЫХ КОЛЬЦАХ

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Аннотация. Пусть $\mathscr{R}$ — первичное кольцо с расширенным центроидом $\mathscr{C}$ и с фактор-кольцо Мартиндейла $\mathscr{Q}$. Аддитивное отображение $\mathscr{F}:\mathscr{R}\to\mathscr{R}$ называют полупроизводной, ассоциированной с $\mathscr{G}:\mathscr{R}\to\mathscr{R}$, если $\mathscr{F}(xy)=\mathscr{F}(x)\mathscr{G}(y)+x\mathscr{F}(y)=\mathscr{F}(x)y+\mathscr{G}(x)\mathscr{F}(y)$ и $\mathscr{F}(\mathscr{G}(x))=\mathscr{G}(\mathscr{F}(x))$ для всех $x,y\in\mathscr{R}$. В этой работе мы исследуем и описываем строение первичных колец $\mathscr{R}$, удовлетворяющих условию $\mathscr{F}(x^m\circ y^n)\in\mathscr{Z}(\mathscr{R})$ для всех $x,y\in\mathscr{R}$, где $m,n\in\mathbb{Z}^+$ и $\mathscr{F}:\mathscr{R}\to\mathscr{R}$ — полупроизоводная с автоморфизмом ξ кольца $\mathscr{R}$. Далее, в качестве приложения нашего теоретико-кольцевого результата мы обсуждаем природу $\mathscr{C}^*$-алгебр. Точнее, для любой примитивной $\mathscr{C}^*$-алгебры $\mathscr{A}$. Точнее, для любой примитивной $\mathscr{C}^*$-алгебры $\mathscr{A}$ получаем следующее. Если антиизоморфизм $\zeta:\mathscr{A}\to\mathscr{A}$ удовлетворяет соотношению $(x^n)^\zeta+x^{n*}\in\mathscr{Z}(\mathscr{A})$ для всех $x,y\in\mathscr{A}$, то $\mathscr{A}$ служит $\mathscr{C}^*-\mathscr{W}_4$-алгеброй, т. е., $\mathscr{A}$ удовлетворяет стандартному тождеству $\mathscr{W}_4(a_1,a_2,a_3,a_4)=0$ for all $a_1,a_2,a_3,a_4\in\mathscr{A}$.

Ключевые слова: первичное кольцо, автоморфизм, полупроизводная.

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