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PARTIAL INTEGRAL OPERATORS OF FREDHOLM TYPE ON KAPLANSKY–HILBERT MODULE OVER L_0

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Dedicated to the 80th anniversary
of Professor Stefan Grigorievich Samko

Abstract. The article studies some characteristic properties of self-adjoint partially integral operators of Fredholm type in the Kaplansky–Hilbert module $L_0\left[L_2\left(\Omega_1\right)\right]$ over $L_0\left(\Omega_2\right)$. Some mathematical tools from the theory of Kaplansky–Hilbert module are used. In the Kaplansky–Hilbert module $L_0\left[L_2\left(\Omega_1\right)\right]$ over $L_0\left(\Omega_2\right)$ we consider the partially integral operator of Fredholm type T_1 (Ω_1 and Ω_2 are closed bounded sets in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, $\nu_1, \nu_2 \in \mathbb{N}$, respectively). The existence of $L_0\left(\Omega_2\right)$ nonzero eigenvalues for any self-adjoint partially integral operator T_1 is proved; moreover, it is shown that T_1 has finite and countable number of real $L_0(\Omega_2)$-eigenvalues. In the latter case, the sequence $L_0(\Omega_2)$-eigenvalues is order convergent to the zero function. It is also established that the operator T_1 admits an expansion into a series of ∇_1-one-dimensional operators.

Key words: partial integral operator, Kaplansky–Hilbert module, L_0-eigenvalue.

Mathematical Subject Classification (2010): 45A05, 47A10, 47G10, 45P05, 45B05, 45C05.

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1. Introduction

Linear equations and operators involving partial integrals appear in elasticity theory, continuum mechanics, aerodynamics and in PDE theory [1]. Self-adjoint partial integral operators arise in the theory of Schrodinger operators [2, 3]. Spectral properties of a discrete Schrodinger operator H are closely related (see [3, 4]) to the partial integral operators which participate in the presentation of operator H.

Let Ω_1 and Ω_2 be closed bounded subsets in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, respectively. Partial integral operator (PIO) of Fredholm type in the space $L_p(\Omega_1 \times \Omega_2)$, $p \geqslant 1$, is an operator of the form [1]

$$T = T_0 + T_1 + T_2 + K, \tag{1}$$

where operators T_0, T_1, T_2 and K are defined by the following formulas

$$T_0 f(x,y) = k_0(x,y) f(x,y),$$

$$T_1 f(x,y) = \int\limits_{\Omega_1} k_1(x,s,y) f(s,y)\, ds, \tag{2}$$

$$T_2 f(x,y) = \int\limits_{\Omega_2} k_2(x,t,y) f(x,t)\, dt,$$

$$K f(x,y) = \int\limits_{\Omega_1}\int\limits_{\Omega_2} k(x,y;s,t) f(s,t)\, ds\, dt.$$

Here k_0, k_1, k_2 and k are given measurable functions on $\Omega_1 \times \Omega_2$, $\Omega_1^2 \times \Omega_2$, $\Omega_1 \times \Omega_2^2$ and $(\Omega_1 \times \Omega_2)^2$, respectively, and all integrals have to be understood in the Lebesgue sense, where $ds = d\mu_1(s)$, $dt = d\mu_2(t)$, $\mu_k(\cdot)$ — the Lebesgue measure on the σ-algebra of subsets Ω_k, $k = 1, 2$.

Furthermore, some simple solvability conditions for the equations $Tf = g$ were investigated by several authors (see, for example, [1] and its references). Spectral properties of the given operator has been studied in [1, 4, 5].

Nevertheless, the description of the spectra of self-adjoint PIOs with L_2 kernels remains an open question. Difficulty of this problem is connected with non-compactness of the operators T_1 and T_2. The article studies some characteristic properties of self-adjoint partially integral operators of Fredholm type in the Kaplansky–Hilbert module $L_0\left[L_2\left(\Omega_1\right)\right]$ over $L_0\left(\Omega_2\right)$. The mathematical tools from the Kaplansky–Hilbert module is used as presented in [6].

The paper is organized as follows. In Section 3 we prove the existence of an L_0-eigenvalue for the PIO T_1.

In Section 4 we study existence of the countable consequence of real L_0-eigenvalues for PIO T_1. In Section 5 it is given the decomposition of the PIO T_1 in series of ∇_1-one-dimensional operators. In Section 5 (in section 6) is given decomposition of the PIO T_1 (the PIO T_2) in series of ∇_1- (∇_2-) one-dimensional operators.

2. Kaplansky–Hilbert Module over L_0

Recall some notions and results from the theory of Kaplansky–Hilbert modules (see [6]).

Let $(\Omega_k, \Sigma_k, \mu_k)$ be a space with complete finite measure μ_k, $L_0(\Omega_k)$-algebra of equivalence classes of all complex measurable functions on $(\Omega_k, \Sigma_k, \mu_k)$, where $k = 1, 2$. We denote by $L_0[L_2(\Omega_1)]$ the set of equivalence classes of all complex measurable functions $f(x,y)$ on $\Omega_1 \times \Omega_2$, which satisfies the condition: the integral

$$\varphi(y) = \int\limits_{\Omega_1} |f(x,y)|^2\, d\mu_1(x)$$

exists for almost all $y \in \Omega_2$ and $\varphi \in L_0(\Omega_2)$.

We consider the map $\langle\cdot,\cdot\rangle_1 : L_0[L_2(\Omega_1)] \times L_0[L_2(\Omega_1)] \to L_0(\Omega_2)$ by rule

$$\langle f, g\rangle_1 = \int\limits_{\Omega_1} f(s,y)\overline{g(s,y)}\, d\mu_1(s).$$

It is clear, that the map $\langle\cdot,\cdot\rangle_1$ satisfies the conditions of $L_0(\Omega_2)$-valued inner product.

For each $f \in L_0[L_2(\Omega_1)]$ we define L_0-norm:

$$\|f\|_1(\omega) = \sqrt{\langle f, f\rangle_1(\omega)}.$$

Then $L_0[L_2(\Omega_1)]$ is Banach–Kantorovich space over $L_0(\Omega_2)$ [6, 7]. Consequently, the space $L_0[L_2(\Omega_1)]$ is Kaplansky–Hilbert module over $L_0(\Omega_2)$ with the inner product $\langle\cdot,\cdot\rangle_1(\omega)$.

If for the map $A : L_0[L_2(\Omega_1)] \to L_0[L_2(\Omega_1)]$ the equality $A(\alpha \cdot f + \beta \cdot g) = \alpha \cdot Af + \beta \cdot Ag$ is hold for all $\alpha, \beta \in L_0(\Omega_2)$, $f, g \in L_0[L_2(\Omega_1)]$, then A is called $L_0(\Omega_2)$-linear operator.

If for the $L_0(\Omega_2)$-linear operator A there exists $C = C(\omega) \in L_0(\Omega_2)$ such that, $\|Af\|_1(\omega) \leqslant C(\omega)\|f\|_1(\omega)$ for all $f \in L_0[L_2(\Omega_1)]$, then A is called $L_0(\Omega_2)$-bounded operator.

For each $L_0(\Omega_2)$-linear $L_0(\Omega_2)$-bounded operator A we define $L_0(\Omega_2)$-norm by the rule

$$\|A\|_1 = \|A\|_1(\omega) = \sup\{\|Af\|_1(\omega) : \ \|f\|_1 \leqslant \mathbf{e}\}.$$

We say the net $(\xi_\alpha)_{\alpha\in A} \subset L_0(\Omega_2)$ (o)-*converges* to the element $\xi \in L_0(\Omega_2)$, whenever there is a decreasing net $(e_\beta)_{\beta\in B} \subset L_0(\Omega_2)$ such that $\inf\{e_\beta : \beta \in B\} = \theta$ and for each $\beta \in B$ there is an index $\alpha(\beta) \in A$ with $|\xi_\alpha - \xi| \leqslant e_\beta$ for all $\alpha \in A : \alpha(\beta) \leqslant \alpha$. In this case, the element ξ is called (o)-limit of the set $(\xi_\alpha)_{\alpha\in A}$ and we write $\xi = (o)\text{-}\lim \xi_\alpha$.

We know [8], that the (o)-converges of the net $(\xi_\alpha)_{\alpha\in A} \subset L_0(\Omega_2)$ to the element ξ is equivalent to converges almost everywhere to the element ξ of the net $(\xi_\alpha)_{\alpha\in A} \subset L_0(\Omega_2)$.

The net $(f_\alpha)_{\alpha\in A}$ in $L_0[L_2(\Omega_1)]$ is called (bo)-converging to $f \in L_0[L_2(\Omega_1)]$, if $(o)\text{-}\lim \|f_\alpha - f\|_1 = \theta$ in $L_0(\Omega_2)$.

Let Λ_2 be the Boolean algebra of idempotents in $L_0(\Omega_2)$. If $(f_\alpha)_{\alpha\in A} \subset L_0[L_2(\Omega_1)]$ and $(\pi_\alpha)_{\alpha\in A}$ is a partition of the unit in Λ_2, then the series $\sum_\alpha \pi_\alpha \cdot f_\alpha$ (bo)-converges in $L_0[L_2(\Omega_1)]$ and its sum is called the *mixing* of $(f_\alpha)_{\alpha\in A}$ *with respect to* $(\pi_\alpha)_{\alpha\in A}$. We denote this sum by $\operatorname{mix}(\pi_\alpha f_\alpha)$. A subset $K \subset L_0[L_2(\Omega_1)]$ is called *cyclic*, if $\operatorname{mix}(\pi_\alpha f_\alpha) \in K$ for each $(f_\alpha)_{\alpha\in A} \subset K$ and any partition of the unit $(\pi_\alpha)_{\alpha\in A}$ in Λ_2. A subset $K \subset L_0[L_2(\Omega_1)]$ is called *cyclically compact*, if K is cyclic and every net in K has a cyclic subset that (bo)-converges to some point of K. A subset is called *relatively cyclically compact*, if it is contained in a cyclically compact set.

A L_0-linear operator in $L_0[L_2(\Omega_1)]$ is called *cyclically compact*, if for every L_0-bounded set B in $L_0[L_2(\Omega_1)]$ the set $A(B)$ is relatively cyclically compact in $L_0[L_2(\Omega_1)]$.

Let T_1 be an operator in the Kaplansky–Hilbert module $L_0[L_2(\Omega_1)]$ over $L_0(\Omega_2)$ given by the formula

$$(T_1 f)(x, y) = \int\limits_{\Omega_1} k_1(x, s, y) f(s, y)\, d\mu_1(s). \tag{3}$$

Here, $k_1(x, s, y)$ is a measurable function on $\Omega_1^2 \times \Omega_2$.

Let the kernel $k_1(x, s, y)$ of the integral operator T_1 satisfy the condition

$$\int\limits_{\Omega_1}\int\limits_{\Omega_1} |k_1(x, s, y)|^2\, d\mu_1(s)\, d\mu_1(x) \in L_0(\Omega_2). \tag{4}$$

Then, the operator T_1 with values in $L_0(\Omega_2)$ is linear and bounded on $L_0[L_2(\Omega_1)]$.

Also, let the kernel $k_1(x, s, y)$ satisfy the condition:

$$k_1(x, s, y) = \overline{k_1(s, x, y)}.$$

Then the operator T_1 is a self-adjoint operator on the Kaplansky–Hilbert module $L_0[L_2(\Omega_1)]$, i. e.,

$$\langle T_1 f, g\rangle_1 = \langle f, T_1 g\rangle_1.$$

A system $\{f_\alpha(x, y)\} \subset L_0[L_2(\Omega_1)]$ is ∇_1-orthogonal system, if $\langle f_\alpha, f_\beta\rangle_1 = \theta$, $\alpha \neq \beta$. A ∇_1-orthogonal system $\{f_\alpha(x, y)\} \subset L_0[L_2(\Omega_1)]$ is said to be ∇_1-orthonormal system, if $\langle f_\alpha, f_\alpha\rangle_1 = \mathbf{e}$.

Note that, the PIO T_1 is a good example for cyclically compact operators on Kaplansky–Hilbert module [7].

3. L_0-Eigenvalue of the Partial Integral Operator T_1

In this section we prove the existence of an L_0-eigenvalue for the PIO. Put $\mathscr{H} = L_0[L_2(\Omega_1)]$.

Theorem 3.1. *The partial integral operator T_1 has non zero L_0-eigenvalue.*

$\lhd$ Put

$$\mathscr{D}_0 = \left\{ \omega \in \Omega_2 : \int\limits_{\Omega_1} \int\limits_{\Omega_1} |k_1(x,s,\omega)|^2 \, d\mu_1(x) \, d\mu_1(s) > 0 \right\}.$$

Then $\mu_2(\mathscr{D}_0) = 0$. For each $f \in \mathscr{H}$, $f \neq \theta$ we define subset $\operatorname{supp}_{\Omega_2}(f)$ with positive measure by the following equality

$$\operatorname*{supp}_{\Omega_2}(f) = \{\omega \in \Omega_2 : \langle f, f \rangle_1(\omega) \neq 0\}.$$

Let $f_0 \in \mathscr{H}$, $\|f_0\|_1(\omega) \neq 0$ for all $\omega \in \mathscr{D}_0$ and $T_1 f_0 \neq \theta$. It is clear, that $T_1^n f_0 \neq \theta$ for all $n \in \mathbb{N}$, as: if

$$T_1^k f_0 \neq \theta, \quad T_1^{k+1} f_0 = \theta, \text{ for some } k \geqslant 1$$

then we get a contradiction

$$\theta = \left\langle T_1^{k+1} f_0, T_1^{k-1} f_0 \right\rangle_1 (\omega) = \left\langle T_1^{k} f_0, T_1^{k} f_0 \right\rangle_1 (\omega) \neq \theta.$$

We construct two sequences $\{\widetilde{f}_k(x,\omega)\}_{k\in\mathbb{N}_0}$, $\{f_k(x,\omega)\}_{k\in\mathbb{N}_0}$ of functions from the Kaplansky–Hilbert module $L_0[L_2(\Omega_1)]$ $(\mathbb{N}_0 = \mathbb{N} \cup 0)$:

$$\widetilde{f}_k(x,\omega) = \begin{cases} \frac{f_k(x,\omega)}{\|f_k\|_1(\omega)}, & x \in \Omega_1,\ \omega \in \operatorname*{supp}_{\Omega_2}(f_k), \\ 0, & x \in \Omega_1,\ \omega \in \Omega_2 \setminus \operatorname*{supp}_{\Omega_2}(f_k), \end{cases}$$

$$f_{k+1}(x,\omega) = (T_1 \widetilde{f}_k)(x,\omega).$$

It follows from [9] that

$$\|f_k\|_1(\omega) \leqslant \|f_{k+1}\|_1(\omega), \quad k \in \mathbb{N}, \tag{5}$$

and

$$\|f_{k+1}\|_1(\omega) \cdot \|f_k\|_1(\omega) = \langle f_{k-1}, f_{k+1} \rangle_1(\omega) = \langle f_{k+1}, f_{k-1} \rangle_1(\omega), \quad k \in \mathbb{N}. \tag{6}$$

On the other hand

$$\|T_1 \widetilde{f}_{k-1}\|_1(\omega) \leqslant \|T_1\|_1(\omega), \quad k \in \mathbb{N},$$

where $\|T_1\|_1(\omega) \in L_0(\Omega_2)$ is the $L_0(\Omega_2)$ valued norm of the PIO T_1. Consequently,

$$\|f_k\|_1(\omega) \leqslant \|T_1\|_1(\omega), \quad k \in \mathbb{N}.$$

Thus, for almost all $\omega \in \Omega_2$ the sequence $\{\|f_k\|_1(\omega)\}_{k\in\mathbb{N}}$ has a finite limit $\lambda(\omega) \geqslant 0$, i. e.,

$$\lim_{k\to\infty} \|f_k\|_1(\omega) = \lambda(\omega), \tag{7}$$

for almost all $\omega \in \Omega_2$. We have $\lambda(\omega) \in L_0(\Omega_2)$, as $\|f_k\|_1(\omega) \in L_0(\Omega_2)$, $k \in \mathbb{N}$. From the relation (5) it follows that $\lambda \neq \theta$. Now, we define the family of integral operators $\{T_1(\omega)\}$ on $L_2(\Omega_1)$ by

$$T_1(\omega)\varphi(x) = \int\limits_{\Omega_1} k_1(x,s,\omega)\varphi(s)\,d\mu_1(s), \quad \varphi \in L_2(\Omega_1),\ \omega \in \Omega_2.$$

Then, $T_1(\omega)$ is a compact operator on $L_2(\Omega_1)$ for almost all $\omega \in \Omega_2$. By the compactness of the operator $T_1(\omega)$ there exists subsequence $\widetilde{f}_{n_i}(x,\omega)$ such that $f_{n_i+1}(x,\omega) = T_1(\omega)\widetilde{f}_{n_i}(x,\omega)$ has a limit $g(x,\omega)$ in the L_0-norm $\|\cdot\|_1$. It is clear $g \in \mathscr{H}$ and $g \neq \theta$. Analogously, for each sequence

$$f_{n_i+2}(x,\omega) = T_1(\omega)\widetilde{f}_{n_i+1}(x,\omega), \quad f_{n_i+3}(x,\omega) = T_1(\omega)\widetilde{f}_{n_i+2}(x,\omega)$$

we obtain $f_{n_i+2} \to h \in \mathscr{H}$ and $f_{n_i+3} \to \widetilde{h} \in \mathscr{H}$ by the L_0-norm $\|\cdot\|_1$.

Using the relations (6), (7) we obtain

$$\|\widetilde{h} - g\|_1^2(\omega) = \lim_{k\to\infty} \|f_{n_k+3} - f_{n_k+1}\|_1^2(\omega)$$
$$= \lim_{k\to\infty} \left\{\|f_{n_k+3}\|_1^2(\omega) + \|f_{n_k+1}\|_1^2(\omega) - \langle f_{n_k+3}, f_{n_k+1}\rangle_1(\omega) - \langle f_{n_k+1}, f_{n_k+3}\rangle_1(\omega)\right\} = 0$$

for almost all $\omega \in \Omega_2$ and so $\widetilde{h} = g$. On the other hand, from the equalities

$$f_{n_k+2}(x,\omega) = \begin{cases} \frac{(T_1(\omega)f_{n_k+1})(x,\omega)}{\|f_{n_k+1}\|_1(\omega)}, & x \in \Omega_1,\ \omega \in \operatorname{supp}\limits_{\Omega_2}(f_{n_k+1}),\\ 0, & x \in \Omega_1,\ \omega \in \Omega_2 \setminus \operatorname{supp}\limits_{\Omega_2}(f_{n_k+1}),\end{cases}$$

$$f_{n_k+3}(x,\omega) = \begin{cases} \frac{(T_1(\omega)f_{n_k+2})(x,\omega)}{\|f_{n_k+2}\|_1(\omega)}, & x \in \Omega_1,\ \omega \in \operatorname{supp}\limits_{\Omega_2}(f_{n_k+2}),\\ 0, & x \in \Omega_1,\ \omega \in \Omega_2 \setminus \operatorname{supp}\limits_{\Omega_2}(f_{n_k+2})\end{cases}$$

we have

$$\|f_{n_k+1}\|_1(\omega) \cdot f_{n_k+2}(x,\omega) = (T_1(\omega)f_{n_k+1})(x,\omega), \quad \omega \in \Omega_2, \tag{8}$$
$$\|f_{n_k+2}\|_1(\omega) \cdot f_{n_k+3}(x,\omega) = (T_1(\omega)f_{n_k+2})(x,\omega), \quad \omega \in \Omega_2. \tag{9}$$

It is clear that

$$\lim_{k\to\infty} \|f_{n_k+1}\|_1(\omega) = \|g\|_1(\omega) = \lim_{k\to\infty} \|f_{n_k+2}\|_1(\omega) = \|h\|_1(\omega)$$
$$= \lim_{k\to\infty} \|f_{n_k+3}\|_1(\omega) = \|\widetilde{h}\|_1(\omega) = \lambda(\omega).$$

From the equalities (8), (9) it follows that

$$\lambda(\omega) \cdot h(x,\omega) = T_1(\omega)g(x,\omega), \quad \lambda(\omega) \cdot \widetilde{h}(x,\omega) = T_1(\omega)h(x,\omega),$$

i. e.,

$$(T_1 g)(x,y) = \lambda(y) \cdot h(x,y), \quad (T_1 h)(x,y) = \lambda(y) \cdot g(x,y).$$

Hence it follows that

$$T_1(h+g)(x,y) = \lambda(y) \cdot (h+g)(x,y), \quad T_1(h-g)(x,y) = -\lambda(y) \cdot (h-g)(x,y).$$

We know, that $h \neq \theta$, $g \neq \theta$. Hence we can conclude that: $h+g \neq \theta$ or $h-g \neq \theta$. It means that the function $\lambda(y)$ is an L_0-eigenvalue of the PIO T_1. $\rhd$

4. Spectral Properties of the Partial Integral Operator T_1 on the Kaplansky–Hilbert Module $L_0[L_2(\Omega_1)]$

Theorem 4.1. *For a PIO T_1 the following function $\lambda_0(\omega) = \sup_{\|g\|_1=\mathbf{e}} |\langle T_1 g, g\rangle_1(\omega)|$ is nonzero and either $+\lambda_0(\omega)$ or $-\lambda_0(\omega)$ is L_0-eigenvalue of the T_1.*

◁ Put

$$\Omega_0 = \left\{ \omega \in \Omega_2 : \int_{\Omega_1}\int_{\Omega_1} |k_1(x,s,\omega)|^2 \, d\mu_1(x)\, d\mu_1(s) > 0 \right\}.$$

From the $T_1 \neq \theta$ it follows that $\lambda_0(\omega) \neq 0$ for all $\omega \in \Omega_0$, i. e., $\lambda_0 \neq \theta$. It is clear, that there is a sequence of ∇_1-normal functions $\{g_n\}_{n=1}^{\infty}$, in which a limit exists

$$(o)\text{-}\lim_{n\to\infty} \langle T_1 g_n, g_n\rangle_1(\omega) = \lambda(\omega),$$

and $\lambda(\omega)$ is a real function on Ω_2, where $\lambda(\omega) = +\lambda_0(\omega)$ or $-\lambda_0(\omega)$. Consequently, $\lambda_0 \in L_0(\Omega_2)$ and $\operatorname{supp}(\lambda) = \Omega_0$.

By cyclical compactness of the PIO T_1 there exists a subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ with

$$(bo)\text{-}\lim_{k\to\infty} (T_1 g_{n_k})(x,y) = h(x,y). \tag{10}$$

Clearly, $\operatorname{supp}_{\Omega_2}(h) = \Omega_0$. From the equality

$$\|T_1 g_{n_k} - \lambda\cdot g_{n_k}\|_1^2 = \|T_1 g_{n_k}\|_1^2 - 2\lambda\cdot\langle T_1 g_{n_k}, g_{n_k}\rangle_1 + \lambda^2$$

we obtain

$$(o)\text{-}\lim_{k\to\infty} \|T_1 g_{n_k} - \lambda\cdot g_{n_k}\|_1^2 = \|h\|_1^2 - \lambda^2. \tag{11}$$

However,

$$\|T_1 g_{n_k}\|_1(\omega) \leqslant \lambda_0(\omega)\cdot\|g_{n_k}\|_1(\omega) = |\lambda(\omega)|.$$

Therefore,

$$\|h\|_1(\omega) \leqslant |\lambda(\omega)|.$$

From this and (11) we have $\|h\|_1(\omega) = |\lambda(\omega)|$. Thus,

$$(o)\text{-}\lim_{k\to\infty} \|T_1 g_{n_k} - \lambda\cdot g_{n_k}\|_1 = \theta. \tag{12}$$

Hence, it follows that

$$T_1 f_0 = \lambda\cdot f_0,$$

where

$$f_0(x,\omega) = \begin{cases} \frac{h(x,\omega)}{\lambda(\omega)}, & x\in\Omega_1,\ \omega\in \operatorname{supp}_{\Omega_2}(\lambda),\\ 0, & x\in\Omega_1,\ \omega\in\Omega_2\setminus \operatorname{supp}_{\Omega_2}(\lambda). \end{cases}$$ ▷

Put

$$\pi_0(\omega) = \begin{cases} 1, & \omega\in \operatorname{supp}_{\Omega_2}(\lambda),\\ 0, & \omega\in\Omega_2\setminus \operatorname{supp}_{\Omega_2}(\lambda). \end{cases}$$

Remark 4.1. Every element $\zeta \in L_0(\Omega_2)$, $\pi_0\zeta = \lambda$ is L_0-eigenvalue of the PIO T_1.

Theorem 4.2. *The PIO T_1 has a finite or countable sequence of ∇_1-orthonormal eigenfunctions*

$$\phi_1(x,y), \phi_2(x,y), \ldots, \phi_n(x,y), \ldots$$

corresponding to a system of real nonzero L_0-eigenvalues

$$\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_n(\omega), \ldots,$$

where

$$|\lambda_1(\omega)| \geqslant |\lambda_2(\omega)| \geqslant \ldots \geqslant |\lambda_n(\omega)| \geqslant \ldots$$

Moreover, for each $f(x,y) \in L_0[L_2(\Omega_1)]$ the equality

$$\|f\|_1^2(\omega) = (o)\text{-}\sum_{k=1}^{\infty} |\langle f, \phi_k \rangle_1(\omega)|^2$$

holds.

$\lhd$ Put $\mathscr{H}_1 = \mathscr{H}$ and $T_1^{(1)} = T_1$. By the Theorem 4.1 there is such element $\phi_1(x,y) \in \mathscr{H}_1$ that $T_1^{(1)}\phi_1 = \lambda_1 \cdot \phi_1$, where λ_1 is a real function on Ω_2 and $\lambda_1(\omega) = \pm \sup_{\|g\|_1 = \mathbf{e}} |\langle T_1 g, g \rangle_1(\omega)|$. We define the Kaplansky–Hilbert submodule $\mathscr{H}_2 = \mathscr{H}_1 \ominus_1 \{\phi_1\}$. It is clear that if $f \in \mathscr{H}_2$, then $T_1^{(1)} f \in \mathscr{H}_2$ from the equality $\langle f, \phi_1 \rangle_1 = \theta$ it follows that

$$\left\langle T_1^{(1)} f, \phi_1 \right\rangle_1 = \langle f, T_1 \phi_1 \rangle_1 = \langle f, \lambda_1 \cdot \phi_1 \rangle_1 = \theta.$$

We define an operator $T_1^{(2)}$ on the $\mathscr{H}_2$ by

$$T_1^{(2)} f = T_1^{(1)} f, \quad f \in \mathscr{H}_2.$$

The operator $T_1^{(2)}$ is a selfadjoint PIO on the $\mathscr{H}_2$. If $T_1^{(2)} \neq \theta$, then we apply Theorem 4.1 to the operator $T_1^{(2)}$ and find an element $\phi_2(x,y) \in \mathscr{H}_2$ such that $T_1^{(2)}\phi_2 = \lambda_2 \cdot \phi_2$, where λ_2 is a real function on Ω_2 and $\lambda_2(\omega) = \pm \sup_{g \in \mathscr{H}_2,\, \|g\|_1 = \mathbf{e}} \left|\left\langle T_1^{(2)} g, g \right\rangle_1(\omega)\right|$. As $\phi_2(x,y) \in \mathscr{H}_2$, $\|\phi_2\|_1 = \mathbf{e}$, we have $\langle \phi_2, \phi_1 \rangle_1 = \theta$. Therefore,

$$|\lambda_2(\omega)| = \sup_{g \in \mathscr{H}_2,\, \|g\|_1 = \mathbf{e}} \left|\left\langle T_1^{(1)} g, g \right\rangle_1(\omega)\right| \leqslant \sup_{g \in \mathscr{H}_1,\, \|g\|_1 = \mathbf{e}} \left|\left\langle T_1^{(1)} g, g \right\rangle_1(\omega)\right| = |\lambda_1(\omega)|.$$

Continuing this process we obtain a sequence of Kaplansky–Hilbert submodules $\mathscr{H}_{k+1} = \mathscr{H}_k \ominus_1 \{\phi_k\}$, where $\phi_k \in \mathscr{H}_k$ are eigenfunctions of the PIO T_1 with $T_1 \phi_k = \lambda_k \cdot \phi_k$.

If $T_1^{(n)}$ is a zero operator for some $n \in \mathbb{N}$ then we obtain the finite system ∇_1-orthonormal eigenfunctions $\phi_1(x,y), \phi_2(x,y), \ldots, \phi_{n-1}(x,y)$ corresponding to the system of nonzero L_0-eigenvalues $\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_{n-1}(\omega)$, such that

$$|\lambda_1(\omega)| \geqslant |\lambda_2(\omega)| \geqslant \ldots \geqslant |\lambda_{n-1}(\omega)|$$

and

$$|\lambda_k(\omega)| = \sup_{g \in \mathscr{H}_k,\, \|g\|_1 = \mathbf{e}} \left|\left\langle T_1^{(1)} g, g \right\rangle_1(\omega)\right|.$$

If $T_1^{(n)} \neq \theta$ for each $n \in \mathbb{N}$ then we obtain an infinite system ∇_1-orthonormal eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ corresponding to the system of L_0-eigenvalues $\lambda_k \neq \theta$. However, the equality

$$T_1(\omega)\phi_k(x,\omega) = \lambda_k(\omega) \cdot \phi_k(x,\omega), \quad k \in \mathbb{N},$$

is correct for almost all $\omega \in \Omega_2$. It follows that $\lim_{k\to\infty} \lambda_k(\omega) = 0$ for almost all $\omega \in \Omega_2$, because $T_1(\omega)$ is a compact operator for almost all $\omega \in \Omega_2$.

Let $f = T_1 h$, $h \in \mathscr{H}$ and $g = h - \sum_{k=1}^{m} \langle h, \phi_k \rangle_1 \cdot \phi_k$. Here m is the number of eigenfunctions of the system $\{\phi_k\}$ when the system $\{\phi_k\}$ is a finite set, and m is equal to arbitrary natural number otherwise. By the equality

$$\langle g, \phi_k \rangle_1 = \theta, \quad k \in \{1, 2, \dots, m\}$$

we have $g \in \mathscr{H}_{m+1}$. Consequently, we have

$$\|T_1 g\|_1^2(\omega) \leqslant \big\|T_1^{(m+1)}\big\|^2(\omega) \cdot \|g\|_1^2(\omega),$$

i. e.,

$$\left\| T_1 h - \sum_{k=1}^{m} \langle h, \phi_k \rangle_1 \cdot T_1 \phi_k \right\|_1^2 (\omega) \leqslant \big\|T_1^{(m+1)}\big\|^2(\omega) \cdot \|g\|_1^2(\omega). \tag{13}$$

We have $\langle h, \phi_k \rangle_1 \cdot T_1 \phi_k = \langle T_1 h, \phi_k \rangle_1 \cdot \phi_k$ and $\|g\|_1 \leqslant \|h\|_1$. Hence by the inequality (13) we obtain

$$\left\| f - \sum_{k=1}^{m} \langle f, \phi_k \rangle_1 \cdot \phi_k \right\|_1^2 (\omega) \leqslant \big\|T_1^{(m+1)}\big\|_1^2(\omega) \cdot \|h\|_1^2(\omega). \tag{14}$$

If the number of elements of the system $\{\phi_k\}$ is equal to m then $T_1^{(m+1)} = \theta$ and we have

$$f = \sum_{k=1}^{m} \langle f, \phi_k \rangle_1 \cdot \phi_k.$$

If the sequence $\{\phi_k\}$ is infinite then from the inequality (14) it follows that

$$\left\| f - \sum_{k=1}^{m} \langle f, \phi_k \rangle_1 \cdot \phi_k \right\|_1^2 (\omega) \leqslant \lambda_{m+1}^2(\omega) \cdot \|h\|_1^2(\omega),$$

i. e.,

$$\theta \leqslant \|f\|_1^2(\omega) - \sum_{k=1}^{m} |\langle f, \phi_k \rangle_1(\omega)|^2 \leqslant \lambda_{m+1}^2(\omega) \cdot \|h\|_1^2(\omega).$$

Thus as $m \to \infty$, we get

$$\|f\|_1^2(\omega) = (o)\text{-}\sum_{k=1}^{\infty} |\langle f, \phi_k \rangle_1(\omega)|^2. \ \triangleright$$

5. Decomposition of the Partial Integral Operator T_1 in Series of ∇_1-One-Dimensional Operators

DEFINITION 5.1. If for an operator $A : \mathscr{H} \to \mathscr{H}$ there are ∇_1-orthonormal functions $\{\phi_k\}_{k=1}^{n} \subset \mathscr{H}$ and some system of functions $\{g_k\}_{k=1}^{n} \subset \mathscr{H}$, such that

$$Af = \sum_{k=1}^{n} \langle f . g_k \rangle_1 \cdot \phi_k, \quad f \in \mathscr{H}$$

then the operator A is called the ∇_1-*n-dimensional operator*, here $\mathscr{H} = L_0[L_2(\Omega_1)]$.

Theorem 5.1. *For the PIO T_1 there is a system of ∇_1-orthonormal functions $\{\phi_k(x,y)\}$ and a sequence of real $L_0(\Omega_2)$-eigenvalues $\lambda_k(\omega)$ such that for all $h \in L_0[L_2(\Omega_1)]$ the following conditions hold:*

1°. $h = h_0 + (bo)\text{-}\sum_{k=1}^{\infty}\langle h, \phi_k\rangle_1 \cdot \phi_k$, $h_0 \in Ker(T_1)$.

2°. $T_1 h = (bo)\text{-}\sum_{k=1}^{\infty} \lambda_k \cdot \langle h, \phi_k\rangle_1 \cdot \phi_k$.

3°. $|\lambda_k(\omega)| \geqslant |\lambda_{k+1}(\omega)|$, $k \in \mathbb{N}$.

4°. $(o)\text{-}\lim_{k\to\infty} \lambda_k = \theta$.

$\triangleleft$ By Theorem 4.2 there are a system of ∇_1-orthonormal functions $\{\phi_k(x,y)\}$ and a sequence of L_0-eigenvalues $\lambda_k(\omega)$ such that $T\phi_k = \lambda_k \cdot \phi_k$ and for each $f = T_1 h$ we get the equality

$$f = (bo)\text{-}\sum_{k=1}^{\infty}\langle f, \phi_k\rangle_1 \cdot \phi_k,$$

where $\langle f, \phi_k\rangle_1 = \lambda_k \cdot \langle h, \phi_k\rangle_1$.

Thus, for all $h \in \mathscr{H}$

$$T_1 h = (bo)\text{-}\sum_{k=1}^{\infty} \lambda_k \cdot \langle h, \phi_k\rangle_1 \cdot \phi_k.$$

If we denote $h_0 = h - (bo)\text{-}\sum_{k=1}^{\infty}\langle h, \phi_k\rangle_1 \cdot \phi_k$, then

$$h = h_0 + (bo)\text{-}\sum_{k=1}^{\infty}\langle h, \phi_k\rangle_1 \cdot \phi_k, \quad T_1 h_0 = \theta.$$

The properties 3° and 4° follows from the Theorem 4.2. Theorem 5.1 can also be proven by using Theorem 3.5 in the article of A. G. Kusraev [10]. $\triangleright$

Theorem 5.2. *For all positive functions $\varepsilon(\omega) \in L_0(\Omega_2)$, $\mu_2(\Omega_2 \setminus \operatorname{supp}(\varepsilon)) = 0$ there exist a ∇_1-finite dimensional operator $\mathscr{T}_1^{\varepsilon}$ on the Kaplansky–Hilbert module $L_0[L_2(\Omega_1)]$, such that $\|T_1 - \mathscr{T}_1^{\varepsilon}\|_1(\omega) < \varepsilon(\omega)$.*

$\triangleleft$ By the Theorem 5.1 there is a system of ∇_1-orthonormal functions $\{\phi_k(x,y)\}$ and a sequence of L_0-eigenvalues $\lambda_k(\omega)$ for which the properties 1°–4° hold. We define the ∇_1-finite dimensional operator $\mathscr{T}_1^{\varepsilon}$:

$$\mathscr{T}_1^{\varepsilon} h = \sum_{k=1}^{n} \lambda_k \cdot \langle h, \phi_k\rangle_1 \cdot \phi_k.$$

It follows that

$$\|T_1 h - \mathscr{T}_1^{\varepsilon} h\|_1^2(\omega) \leqslant \lambda_{n+1}^2(\omega) \cdot \left\{|\langle h, \phi_{k+1}\rangle_1(\omega)|^2 + |\langle h, \phi_{k+2}\rangle_1(\omega)|^2 + \ldots\right\} \leqslant \lambda_k^2(\omega)\|h\|_1^2(\omega).$$

Hence, for $|\lambda_{n+1}(\omega)| < \varepsilon(\omega)$ we have $\|T_1 - \mathscr{T}_1^{\varepsilon}\|_1(\omega) < \varepsilon(\omega)$. $\triangleright$

6. Decomposition of the Partial Integral Operator T_2 in Series of ∇_2-One-Dimensional Operators

We denote by $L_0[L_2(\Omega_2)]$ the set of equivalence classes of all complex measurable functions $f(x,y)$ on $\Omega_1 \times \Omega_2$, which satisfied the condition: the integral

$$\psi(x) = \int_{\Omega_2} |f(x,y)|^2 \, d\mu_2(y)$$

exist for almost all $x \in \Omega_1$ and $\psi \in L_0(\Omega_1)$.

We define $L_0(\Omega_1)$-valued inner product on $L_0[L_2(\Omega_2)]$ by

$$\langle f, g\rangle_2 = \int\limits_{\Omega_1} f(x,t)\overline{g(x,t)}\, d\mu_2(t).$$

For each $f \in L_0[L_2(\Omega_2)]$ we define L_0-norm: $\|f\|_2(\upsilon) = \sqrt{\langle f, f\rangle_2(\upsilon)}$. Then $L_0[L_2(\Omega_2)]$ is a Banach–Kantorovich space over $L_0(\Omega_1)$. Consequently, the space $L_0[L_2(\Omega_2)]$ is a Kaplansky–Hilbert module over $L_0(\Omega_1)$ with the inner product $\langle\cdot,\cdot\rangle_2(\upsilon)$.

Let T_2 be an operator in the Kaplansky–Hilbert module $L_0[L_2(\Omega_2)]$ over $L_0(\Omega_1)$ given by the formula

$$(T_2 f)(x,y) = \int\limits_{\Omega_2} k_2(x,t,y) f(x,t)\, d\mu_2(t). \tag{15}$$

Here, $k_2(x,t,y)$ is measurable function on $\Omega_1 \times \Omega_2^2$.

Assume that the kernel $k_2(x,s,y)$ of the integral operator T_2 satisfies the condition

$$\int\limits_{\Omega_2}\int\limits_{\Omega_2} |k_2(x,t,y)|^2\, d\mu_2(t)\, d\mu_2(y) \in L_0(\Omega_1).$$

Then, the operator T_2 is linear and $L_0(\Omega_1)$-bounded operator on $L_0[L_2(\Omega_2)]$. If the kernel $k_2(x,s,y)$ satisfy of the condition $k_2(x,t,y) = \overline{k_2(x,y,t)}$, then the operator T_2 is a self-adjoint operator on the Kaplansky–Hilbert module $L_0[L_2(\Omega_2)]$, i. e.,

$$\langle T_2 f, g\rangle_2 = \langle f, T_2 g\rangle_2.$$

A system $\{f_\alpha(x,y)\} \in L_0[L_2(\Omega_2)]$ is said ∇_2-orthogonal system, if $\langle f_\alpha, f_\beta\rangle_2 = \theta$, $\alpha \neq \beta$. A ∇_2-orthogonal system $\{f_\alpha(x,y)\} \subset L_0[L_2(\Omega_2)]$ is said ∇_2-orthonormal system, if $\langle f_\alpha, f_\alpha\rangle_2 = \mathbf{e}$.

Note that, the PIO T_2 is cyclically compact on the Kaplansky–Hilbert module $L_0[L_2(\Omega_2)]$ [7].

Theorem 6.1. *For the PIO T_2 there is a system of ∇_2-orthonormal functions $\{\psi_k(x,y)\}$ and a sequence of real $L_0(\Omega_1)$-eigenvalues $\zeta_k(\upsilon)$ such that, for all $h \in L_0[L_2(\Omega_2)]$ the following hold:*

1°. $h = h_0 + (bo)\text{-}\sum_{k=1}^{\infty}\langle h, \psi_k\rangle_1 \cdot \psi_k$, $h_0 \in Ker(T_2)$;
2°. $T_2 h = (bo)\text{-}\sum_{k=1}^{\infty}\zeta_k \cdot \langle h, \psi_k\rangle_1 \cdot \psi_k$, *where*
3°. $|\zeta_k(\upsilon)| \geqslant |\zeta_{k+1}(\upsilon)|$, $k \in \mathbb{N}$;
4°. $(o)\text{-}\lim_{k\to\infty} \zeta_k = \theta$.

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ЧАСТИЧНО ИНТЕГРАЛЬНЫЕ ОПЕРАТОРЫ ТИПА ФРЕДГОЛЬМА В МОДУЛЕ КАПЛАНСКОГО — ГИЛЬБЕРТА НАД L_0

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Аннотация. В статье изучаются некоторые характеристические свойства самосопряженных частично интегральных операторов типа Фредгольма в модуле Капланского — Гильберта $L_0\left[L_2\left(\Omega_1\right)\right]$ над $L_0\left(\Omega_2\right)$. Используется математический инструментарий из теории модулей Капланского — Гильберта. В модуле Капланского — Гильберта $L_0\left[L_2\left(\Omega_1\right)\right]$ над $L_0\left(\Omega_2\right)$ рассматриваются частично интегральные операторы типа Фредгольма T_1 (Ω_1 и Ω_2 — замкнутые ограниченные множества в $\mathbb{R}^{\nu_1}$ и $\mathbb{R}^{\nu_2}$, $\nu_1, \nu_2 \in \mathbb{N}$ соответственно). В работе доказано существование $L_0\left(\Omega_2\right)$-собственных значений, отличных от нуля для любого самосопряженного частично интегрального оператора типа Фредгольма T_1; более того, показано существование конечного или счетного числа вещественных $L_0\left(\Omega_2\right)$-собственных значений. В последнем случае, последовательности $L_0\left(\Omega_2\right)$-собственных значений порядково сходятся к нулевой функции. Установлена также теорема о разложимости оператора T_1 в ряд по ∇_1 одномерным операторам.

Ключевые слова: частично интегральный оператор, модуль Капланского — Гильберта, L_0-собственное значение.

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