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ON THE STRUCTURE OF ARCHIMEDEAN f-RINGS[#]

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Abstract. It is proved that the Boolean valued representation of a Dedekind complete f-ring is either the group of integers with zero multiplication, or the ring of integers, or the additive groups of reals with zero multiplication, or the ring of reals. Correspondingly, the Dedekind completion of an Archimedean f-ring admits a decomposition into the direct sum of for polars: singular ℓ-group and an erased vector lattice, both with zero multiplication, a singular f-rings and an erased f-algebra. A corollary on a functional representation of universally complete f-rings is also given.

Key words: vector lattice, f-ring, f-algebra, Boolean valued representation, singular f-ring.

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A *lattice-ordered ring*, denoted ℓ-ring, is a ring R whose underlying set is lattice-ordered such that $(R, \leqslant, +)$ is an ℓ-group and such that if $a \leqslant b$ and $0 \leqslant c$ in R, then $ac \leqslant bc$ and $ca \leqslant cb$. An *ℓ-algebra* is an algebra over the reals whose underlying vector space and ring are vector lattice and ℓ-ring, respectively. An ℓ-ring (ℓ-algebra) R is called an *f-ring* (*f-algebra*) if $x \wedge y = 0$ implies $ax \wedge y = 0$ and $xa \wedge y = 0$ for all $x, y \in E$ and $a \in R_+$ (see [1, 2]). By *erased vector lattices* (*erased f-algebras*) we mean the ℓ-groups (f-rings) that result from vector lattices (f-algebras) by ignoring the multiplication by real numbers, see [4].

Say that the elements $x, y \in G$ of an ℓ-group are *disjoint* and write $x \perp y$ if $|x| \wedge |y| = 0$. For a nonempty subset $M \subset G$ the *polar* $M^{\perp}$ is defined as $M^{\perp} := \{x \in G :\ (\forall\, y \in M)\, x \perp y\}$. The inclusion ordered set of all polars in G is a complete Boolean algebra denoted by $\mathbb{P}(G) := (\mathbb{P}(G), \vee, \wedge, *)$, where $L \wedge M = L \cap M$, $L \vee M = (L \cup M)^{\perp\perp}$, $L^* = L^{\perp}$ $(L, M \in \mathbb{P}(G)$. If for every polar $L \in \mathbb{P}(G)$ there is a *polar decomposition* $G = L \oplus L^{\perp}$ then G is called *strongly projectable*; polar projections form a complete Boolean algebra which is isomorphic to $\mathbb{P}(G)$ and denoted by the same symbol, see [1, 2] for more details.

For a complete Boolean algebra $\mathbb{B}$, denote by $\mathbb{V}^{(\mathbb{B})}$ the corresponding Boolean valued model of set theory, [3, chap. 2]. There is a natural way of assigning to each statement ϕ about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ the *Boolean truth-value* $[\![\phi(x_1, \dots, x_n)]\!] \in \mathbb{B}$. The sentence $\phi(x_1, \dots, x_n)$ is called *true within* $\mathbb{V}^{(\mathbb{B})}$ if $[\![\phi(x_1, \dots, x_n)]\!] = \mathbb{1}$. For every complete Boolean algebra $\mathbb{B}$, all the theorems of ZFC are true in $\mathbb{V}^{(\mathbb{B})}$. The *ascending-and-descending machinery* providing an

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interaction between the Boolean-valued universe $\mathbb{V}^{(\mathbb{B})}$ and the world of ordinary sets, rests on the functors of *canonical embedding* $X \mapsto X^\wedge$ and *ascent* $X \mapsto X\uparrow$, both acting from $\mathbb{V}$ to $\mathbb{V}^{(\mathbb{B})}$, and the functor of *descent* $X \mapsto X\downarrow$, acting from $\mathbb{V}^{(\mathbb{B})}$ to $\mathbb{V}$; see [3] for details. Boolean valued technology enables one to reduce some problems concerning ℓ-groups and f-rings to that of linearly ordered groups and rings, see [3, § 4.4]. The aim of this note is to outline a Boolean values approach to the structure theory of f-rings. Denote by $\mathscr{R}$ the fields of reals within $\mathbb{V}^{(\mathbb{B})}$. Then the *descent* $\mathscr{R}\downarrow$ is a universally complete vector lattice, [3, 5.2.2]. Below, $\mathbb{Z}$ will denote the integers and $\mathbb{R}$ the reals, both with the usual addition, multiplication, and order; denote by $\mathbb{Z}_a$ and $\mathbb{R}_a$ the additive grope of $\mathbb{Z}$ and the ring of $\mathbb{R}$, respectively. Note that $\mathbb{Z}^\wedge$ is the ring of integers within $\mathbb{V}^{(\mathbb{B})}$ and $(\mathbb{Z}_a)^\wedge = (\mathbb{Z}^\wedge)_a$, whilst the equality $\mathbb{R}^\wedge = \mathscr{R}$ amounts to the σ-distributivity of $\mathbb{B}$.

Lemma 1. *Let K be a nonzero order complete f-ring, $\mathbb{B} = \mathbb{P}(K)$ and $\mathscr{K}$ the Boolean valued representation of K in the model $\mathbb{V}^{(\mathbb{B})}$. Then $\mathbb{V}^{(\mathbb{B})} \models$"$\mathscr{K}$ is either the group of integers with zero multiplication $\mathbb{Z}_a^\wedge$, or the ring of integers $\mathbb{Z}^\wedge$, or the additive groups of reals with zero multiplication $\mathscr{R}_a$, or the ring of reals $\mathscr{R}$".*

$\lhd$ Sketch of the Proof. According to [3, Theorem 4.4.13] $\mathscr{K}$ is either a subgroup $\mathscr{K}_0$ of $\mathscr{R}_a$ (with zero multiplication) or a subring $\mathscr{K}_1$ of $\mathscr{R}$. Since K is order complete, so is $\mathscr{K}$ by [3, Theorem 4.4.10(2)]. It follows that the subgroup $\mathscr{K}_0$ is either $\mathbb{Z}_a$ or is $\mathscr{R}_a$, whilst the subring $\mathscr{K}_1$ is either $\mathbb{Z}$ or is $\mathscr{R}$. $\rhd$

Lemma 2. *Let K is an Archimedean f-ring and $\mathscr{K}$ be its Boolean valued representation of K^δ. There are a Boolean isomorphism ι from $\mathbb{B}$ onto $\mathbb{P}(K^\delta)$ and a complete monomorphism j from K into $K' := \mathscr{K}\downarrow$ such that $b \leqslant [\![0 \leqslant j(g)]\!] \Longleftrightarrow 0 \leqslant \iota(b)j(g)$ for all $g \in G'$ и $b \in \mathbb{B}$.*

$\lhd$ Lemma 1 is applicable to the Dedekind completion K^δ. Combining this with [3, Theorem 4.4.10] we arrive at the desired result. $\rhd$

A ℓ-ring K is said to be *laterally complete* if each its disjoint subset has the least upper bound. Each ℓ-ring K has a uniquely determined *lateral completion* denoted by K^λ, [4]. Say that K is universally complete if $K = (K^\delta)^\lambda$.

Corollary 1. *$K' = (j(K)^\delta)^\lambda$, i. e., (K', j) is the universal completion of K.*

An element $0 < g$ in an ℓ-group G is *singular* if $0 \leqslant h \leqslant g$ implies $h \wedge (g - h) = 0$ for all $h \in G$ or, equivalently, if the order interval $[0, x]$ of G is a Boolean algebra. An ℓ-group G is a *singular ℓ-group* if for each $0 < g \in G$, there exists a singular element $s \in G$ such that $0 < s \leqslant g$. A *singular f-ring* is an f-ring whose underlying ℓ-group is singular. Denote by $C_\infty(Q, \mathbb{F})$ a part of $C_\infty(Q, \mathbb{R})$ (see [1, p. 127] for definition) consisting of function with values in $\mathbb{F}$, where $\mathbb{F} := \mathbb{Z}_a, \mathbb{Z}, \mathbb{R}_a, \mathbb{R}$. The next two results can be derived making use of [3, Theorems 4.4.12 and 4.4.13] and [5, Theorem 3.5].

Theorem. *Let K be an Archimedean f-ring and $\mathbb{B} = \mathbb{P}(K)$. There is a polar decomposition $K^\delta = G_s \oplus G_r \oplus H_s \oplus H_r$ where G_s and G_r are respectively a singular ℓ-group and an erased vector lattice, both with zero multiplication, while H_s and H_r are respectively a singular f-ring and an f-algebra with erased scalar multiplication.*

$\lhd$ Sketch of the Proof. The proof follows the same lines as the proof of [3, Theorem 3.5]. Put $b_s = [\![\mathscr{K} = \mathbb{Z}_a^\wedge]\!]$, $b_r = [\![\mathscr{K} = \mathscr{R}_a]\!]$, $d_s = [\![\mathscr{K} = \mathbb{Z}^\wedge]\!]$, and $d_r = [\![\mathscr{K} = \mathscr{R}]\!]$. By Lemma 1 we observe that $b_s, b_r, d_s, d_r \in \mathbb{B}$ are pairwise disjoint and $b_s \vee b_r \vee d_s \vee d_r = \mathbb{1}$. Arguing as in [4, Theorem 3.5] we define $\mathscr{G}_s := b_s \wedge \mathbb{Z}_a^\wedge$, $\mathscr{H}_s := d_s \wedge \mathbb{Z}^\wedge$, $\mathscr{G}_r := b_r \wedge \mathscr{R}_a$, and $\mathscr{H}_r := d_r \wedge \mathscr{R}$. Now the f-rings $G_s := K^\delta \cap \mathscr{G}_s\downarrow$, $G_r := K^\delta \cap \mathscr{G}_r\downarrow$, $H_s := K^\delta \cap \mathscr{H}_s\downarrow$, and $H_r := K^\delta \cap \mathscr{H}_r\downarrow$ are the desired polars of K^δ. $\rhd$

Corollary 2. *Let K be an Archimedean f-ring, $\mathbb{B}=\mathbb{P}(K)$, and Q the Stone representation space of $\mathbb{B}$. Then there exist clopen sets Q_k $(k=1,2,3,4)$ in Q such that $Q=\bigcup_{k=1}^{4} Q_k$ and*

$$K^{\delta\lambda}\simeq C_\infty(Q_1,\mathbb{Z}_a)\oplus C_\infty(Q_2,\mathbb{Z})\oplus C_\infty(Q_3,\mathbb{R}_a)\oplus C_\infty(Q_4,\mathbb{R}).$$

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О СТРОЕНИИ АРХИМЕДОВЫХ f-КОЛЕЦ

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Аннотация. Установлено, что булевозначное представление порядково полного f-кольца представляет собой либо группу целых чисел с нулевым умножением, либо кольцо целых чисел, либо аддитивнуюгруппу поля действительных чисел с нулевым умножением, либо кольцо действительных чисел. Соответственно, порядковое пополнение архимедова f-кольцо допускает разложение в прямую сумму четырех поляр: ℓ-группы и стертой векторной решетки, обе с нулевым умножением, сингулярного f-кольца и стертой f-алгебры. Приводится также следствие о функциональном представлении универсально полных f-колец.

Ключевые слова: векторная решетка, f-кольцо, f-алгебра, булевозначная модель, сингулярное f-кольцо.

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