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PLURIHARMONIC DEFINABLE FUNCTIONS IN SOME O-MINIMAL EXPANSIONS OF THE REAL FIELD

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Abstract. In this paper, we first try to solve the following problem: If a pluriharmonic function f is definable in an arbitrary o-minimal expansion of the structure of the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, -, ., 0, 1, <)$, is this function locally the real part of a holomorphic function which is definable in the same expansion? In Proposition 2.1 below, we prove that this problem has a positive answer if the Weierstrass division theorem holds true for the system of the rings of real analytic definable germs at the origin of $\mathbb{R}^n$. We obtain the same answer for an o-minimal expansion of the real field which is pfaffian closed (Proposition 2.6) for the harmonic functions. In the last section, we are going to show that the Weierstrass division theorem does not hold true for the rings of germs of real analytic functions at $0 \in \mathbb{R}^n$ which are definable in the o-minimal structure $(\overline{\mathbb{R}}, x^{\alpha_1}, \dots, x^{\alpha_p})$ where $\alpha_1, \dots, \alpha_p$ are irrational real numbers.

Key words: o-minimal structures, pluriharmonic function, Weierstrass division theorem.

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1. Introduction

The set of harmonic functions on a given open set $U \subset \mathbb{R}^n$ can be seen as the kernel of the Laplace operator Δ, and is therefore a vector space over $\mathbb{R}$: sums, differences and scalar multiples of harmonic functions are again harmonic. If f is a harmonic function on U, then all partial derivatives of f are also harmonic functions on U. The Laplace operator Δ and the partial derivative operator will commute on this class of functions. These functions are closely connected to holomorphic maps since the real and imaginary parts of a holomorphic function are harmonic functions. The study of harmonic functions is important in physics and engineering and there are many results in the theory of harmonic functions that are not connected directly with complex analysis.

A real valued pluriharmonic function is a function whose restriction to every complex line is a harmonic function with respect to the real and to the imaginary part of the complex line parameter.

We recall by [1, Chapter 7.1.4] that every harmonic function is locally the real part of a holomorphic function, whereas, in finitely many complex variables every pluriharmonic function (of real valued) is locally the real part of a holomorphic function of several complex variables [2, Theorem 26]. Every pluriharmonic function is a harmonic function.

Suppose that $\mathscr{R}$ is an o-minimal expansion of the structure $\overline{\mathbb{R}} := (\mathbb{R}, +, -, \cdot, 0, 1, <)$. The Pfaffian closure of $\mathscr{R}$ is the smallest expansion of $\mathscr{R}$ that is closed under taking Rolle leaves of definable $\mathscr{C}^1$ forms, it is o-minimal (see [3] for precise definitions and results).

It is well known in the classical analysis that every harmonic function is locally the real part or the imaginary part of a holomorphic function. In this paper we first show by using a result of T. Kaiser in [4] that if, a pluriharmonic function is definable in an arbitrary o-minimal expansion of the real field which satisfies the Weierstrass division theorem for the rings of germs of real analytic definable functions, then this function is locally the real part of a holomorphic definable function, also we obtain the same result for an o-minimal expansion of the real field, which is Pfaffian closed.

The Weierstrass division theorem is the key tool for local complex analytic geometry (see for example Gunning and Rossi [5, Chapter II]). In section 3, we will show that the Weierstrass division theorem does not hold true over the o-minimal structure $(\overline{\mathbb{R}}, x^{\alpha_1}, \ldots, x^{\alpha_p})$ (where $\alpha_1, \ldots, \alpha_p$ are irrational real numbers) for the rings of germs of real analytic functions at $0 \in \mathbb{R}^n$ which are definable in this structure.

2. Definability of a Holomorphic Function in Some o-Minimal Structures

We know that every pluriharmonic function f is locally the real part of a holomorphic function. In this section, we tackle the question that if we take this function f to be definable in an o-minimal expansion of the real field, is this function locally the real part of a holomorphic definable function?

In this section, we will give some answers to this question.

Let $\mathscr{R}$ be an arbitrary o-minimal expansion of the structure $\overline{\mathbb{R}} := (\mathbb{R}, +, -, \cdot, 0, 1, <)$. "Definable" will always mean "definable with parameters".

We denote by $\mathscr{C}^{\omega}_{\mathscr{R},n}$ (where $n \in \mathbb{N}$) the rings of germs of functions that are real analytic on a neighbourhood of $0 \in \mathbb{R}^n$, and definable in the structure $\mathscr{R}$.

A germ $f \in \mathscr{C}^{\omega}_{\mathscr{R},n}$ is called regular in x_n of order p with respect to the variable x_n if, there exists a germ $h \in \mathscr{C}^{\omega}_{\mathscr{R},n}$ such that $f = x_n^p h(0, x_n)$ with $h(0) \neq 0$.

We say that the system $\{\mathscr{C}^{\omega}_{\mathscr{R},n}; n \in \mathbb{N}\}$ satisfies Weierstrass division theorem if $f, g \in \mathscr{C}^{\omega}_{\mathscr{R},n}$ such that the germ g is regular of order p with respect to the variable x_n, there exist $q \in \mathscr{C}^{\omega}_{\mathscr{R},n}$ and $r_1, \ldots, r_p \in \mathscr{C}^{\omega}_{\mathscr{R},n-1}$ such that

$$f = gq + \sum_{j=1}^{p} r_j(x_1, \ldots, x_{n-1}) x_n^{p-j}.$$

We say that the structure $\mathscr{R}$ has complexification if, the Weierstrass division theorem holds true over for the system $(\mathscr{C}^{\omega}_{\mathscr{R},n})_{n \in \mathbb{N}}$.

Problem (1): If a pluriharmonic function is definable in an arbitrary o-minimal expansion of the structure $\overline{\mathbb{R}}$, is this function locally the real part of a holomorphic definable function?

Unfortunately, the answer is negative for the given o-minimal structure $\mathscr{R} := (\overline{\mathbb{R}}, \exp)$. In fact, let's take the harmonic function defined on $\mathbb{R}^2 \backslash \{0\}$ as follows $f : (x, y) \to \ln(x^2 + y^2)$. Every point $(a, b) \neq (0, 0)$ admits a neighborhood D where the function f is harmonic and definable, so if it is the real part of a holomorphic definable function F, the imaginary part of the function F denoted g is also definable in the structure $\mathscr{R}$, by [6, Theorem 4], these functions f and g are also definable in $\overline{\mathbb{R}}$, which is a contradiction as f is not an algebraic function.

Under what conditions does Problem (1) admit a positive answer? which is the aim of the sequel of this section.

Proposition 2.1. *If the Weierstrass division theorem holds true for the system* $(\mathscr{C}^{\omega}_{\mathscr{R},n})_{n\in\mathbb{N}}$, *then every harmonic* (*pluriharmonic in more than two real variables*) *definable function* f *is locally the real part of a holomorphic definable function.*

$\lhd$ It suffices to prove this proposition for the harmonic functions, let f be a harmonic definable function, so it is locally the real part of a holomorphic function F. We use Ahlfors' trick [7, Chapter 2, Section 2.1], so then substituting the variable z by $x+iy$, we obtain the power series $F_1(x,y)=F(x+iy)\in\mathbb{C}[[x,y]]$. Then

$$f(x,y)=\frac{F(x+iy)+\overline{F(x+iy)}}{2}\in\mathbb{C}[[x,y]].$$

We now substitute $z/2$ for x and $z/2i$ for y in the series $f(x,y)$, obtaining the series $F_2(z)=f(z/2,z/2i)$. Thus we have

$$F_2(z)=f(z/2,z/2i)=\frac{F(z)+\overline{F(0)}}{2},$$

that is, $F(z)=2f(z/2,z/2i)-\overline{F(0)}$.

As the function f is real analytic and definable, so by [4, Theorem 2.4], the function $f(z,z')$ is definable and the function $z\to(z/2,z/2i)$ is definable, we deduce that the function $z\to f(z/2,z/2i)$ is also definable and so is the function F. $\rhd$

EXAMPLE 2.1. For the structure $\mathscr{R}:=\overline{\mathbb{R}}$, it is well known thanks to [8, Theorem 8.2.9] that the system $(\mathscr{C}^{\omega}_{\mathscr{R},n})_{n\in\mathbb{N}}$ satisfies the Weierstrass division theorem, so every harmonic definable function is locally the real part of a holomorphic definable function.

For a holomorphic function F whose real and imaginary parts are respectively f and g, we observe that with the same reasoning we obtain $F(z)=2ig(z/2,z/2i)+\overline{F(0)}$. We deduce the following proposition:

Proposition 2.2. *If the structure* $\mathscr{R}$ *has complexification, then a holomorphic function is definable if and only if either its real part or its imaginary part is definable.*

DEFINITION 2.1. Let $U\subset\mathbb{R}^n$. A sequence $f_1,\dots,f_r\in\mathscr{C}^1(U)$ is called a Pfaffian chain, if there exist polynomials $p_{ij}(x_1,\dots,x_n,y_1,\dots,y_i)$ for $i=1,\dots,r$ and $j=1,\dots,n$, such that for all $x\in U$

$$\frac{\partial f_i}{\partial x_j}(x_1,\dots,x_n)=p_{i,j}\big(x_1,\dots,x_n,f_1(x_1,\dots,x_n),\dots,f_i(x_1,\dots,x_n)\big).$$

We say that a function $f\in\mathscr{C}^1(U)$ is Pfaffian if,

$$f(x_1,\dots,x_n)=q\big(x_1,\dots,x_n,f_1(x_1,\dots,x_n),\dots,f_r(x_1,\dots,x_n)\big)$$

for some polynomial q and some Pfaffian chain $f_1,\dots,f_r$ on U.

We recall that the Pfaffian closure of an o-minimal structure $\mathscr{R}$ (expansion of $\overline{\mathbb{R}}$) is the smallest expansion $\mathscr{P}$ of $\mathscr{R}$ such that every function which is Pfaffian over $\mathscr{P}$ is already definable in $\mathscr{P}$ (with parameters).

EXAMPLE 2.2. We define $f_1(x)=\exp(x)$ and $f_{m+1}(x)=\exp(f_m(x))$. Then $f'_m=f_1f_2\dots f_m$, so the sequence $f_1,\dots,f_m$ is a Pfaffian chain.

Proposition 2.3. *Let $\mathscr{R}$ be an o-minimal expansion of the real field. If the structure $\mathscr{R}$ is pfaffian closed, then every harmonic definable function f is locally the real part of a holomorphic definable function.*

$\lhd$ We know that in a vicinity of a given point the function f is the real part of a holomorphic function $F = f + ig$, where f and g are respectively the real and the imaginary parts of F.

Since f is definable, so are both partial derivatives of f (as two variables functions). By the Cauchy–Riemann equations, it follows that the partial derivatives of g are also definable in $\mathscr{R}$ (as two variables functions). So g is pfaffian over $\mathscr{R}$ (as two variables functions). Since $\mathscr{R}$ is pfaffian closed, it follows that g is definable.

Therefore the function F is definable in the structure $\mathscr{R}$. $\rhd$

Remark 2.1. We deduce from Proposition 2.1 that the converse of Proposition 2.3 is not true. In fact, by [4, Theorem 4.4] the structure $\overline{\mathbb{R}} := (\mathbb{R}, +, -, \, .\, , 0, 1, <)$ has complexification, so thanks to Proposition 2.1, every harmonic definable function is locally the real part of a holomorphic definable function, but the structure $\overline{\mathbb{R}}$ is not pfaffian closed as the exp function is a pfaffian function, but it is not definable in this structure.

Conclusion (1): If the reciprocal of Proposition 2.1 holds true, then the pfaffian closure of every o-minimal expansion of the real field satisfies the Weierstrass division theorem for the ring of real analytic definable germs.

3. Weierstrass Division Theorem over the Structure $\mathscr{M}$

As mentioned in the introduction, we let $\mathscr{M}$ denote the o-minimal structure $(\overline{\mathbb{R}}, x^{\alpha_1}, \dots, x^{\alpha_p})$, where $\alpha_1, \dots, \alpha_p$ is a finite sequence of irrational real numbers and by x^{α_i} the function which is equal to the usual x^{α_i} if $x > 0$ and equals to 0, if $x \leqslant 0$ for all $i = 1, \dots, p$. This structure is studied in detail in [9].

The aim of this section is to prove the failure of the Weierstrass division theorem for the ring of real analytic definable germs in the structure $\mathscr{M}$.

Theorem 3.1. *The system of the rings of real analytic germs at $0 \in \mathbb{R}^n$ that are definable in the structure $\mathscr{M}$ does not satisfy the Weierstrass division theorem.*

$\lhd$ Assume that the function $F : z \mapsto (z+1)^{\alpha_1}$ restricted to some small disc centered at the origin is definable in $\mathscr{M}$. So the real part and the imaginary part of the function F are definable in the structure $\mathscr{M}$, consequently the function $f : x \mapsto (x+1)^{\alpha_1}$ restricted to some open interval around zero is definable in $\mathscr{M}$, then by Corollary 1 in Section 4 of [6], the germ of the function $f : x \mapsto (x+1)^{\alpha_1}$ is definable in the structure $\overline{\mathbb{R}}$. But this is a contradiction to the fact that a power function with irrational exponent is not algebraic (see [9]). Suppose that the structure $\mathscr{M}$ satisfies the Weierstrass division theorem for the rings of the real analytic definable germs, f is a real analytic definable germ in the structure $\mathscr{M}$, by [4, Theorem 2.4] the germ of the function F is also definable in the structure $\mathscr{M}$, which is a contradiction. $\rhd$

We end this paper by this interesting remark that allows us to prove Theorem 3.1 without using the result described in [4, Theorem 2.4].

Remark 3.1. Let $z \in \mathbb{R}$, set $g = (z+1)^{\alpha_1}$ and $f = z^2 - 2xz + (x^2 + y^2)$. It is clear that the germ g is real analytic (at the origin) and definable in the structure $\mathscr{M}$. Suppose that the Weierstrass division theorem holds true over the structure $\mathscr{M}$ for the ring of real analytic definable functions germs at $0 \in \mathbb{R}^n$ and let's devide g by f to obtain

$$(z+1)^{\alpha_1} = \left[z^2 - 2xz + \left(x^2 + y^2\right)\right] Q(x, y, z) + R_0(x, y) + z R_1(x, y),$$

where the functions Q, R_0 and R_1 are definable in $\mathscr{M}$, set $z = x + iy$, we have $(x+1+iy)^{\alpha_1} = R_0(x,y) + xR_1(x,y) + iyR_1(x,y)$ and the real part of the function $(x+1+iy)^{\alpha_1}$ is the real function $R_0(x,y) + xR_1(x,y)$ and its imaginary part is the function $yR_1(x,y)$. So the function $(x+1+iy)^{\alpha_1}$ is also definable in the structure $\mathscr{M}$. But by the proof of Theorem 3.1, we get that this function germ is not definable in the structure $\mathscr{M}$, which is a contradiction.

Conclusion (2): Thanks to Remark 3.1, we have constructed an example where the quotient and the remainder of this division do exist but they are not definable in anywhere.

Problem (2): Let's take a harmonic (pluriharmonic in more than two real variables) definable function in the structure $\mathscr{M}$, is this function algebraic (i. e., definable in the structure $\overline{\mathbb{R}}$)?

In case we have a positive response to Problem (2), we will have built a structure for which Problem (1) has a positive answer, but this structure does not satisfy the Weierstrass division theorem for the rings of real analytic definable germs, and therefore, we will have proved that the reciprocal of the above Proposition 2.1 does not hold true.

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ПЛЮРИГАРМОНИЧЕСКИЕ ОПРЕДЕЛИМЫЕ ФУНКЦИИ В НЕКОТОРЫХ o-МИНИМАЛЬНЫХ РАСШИРЕНИЯХ ВЕЩЕСТВЕННОГО ПОЛЯ

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Аннотация. В настоящей статье предпринимается попытка решить следующую задачу: если плюригармоническая функция f определима в произвольном o-минимальном расширении структуры вещественного поля $\overline{\mathbb{R}} := (\mathbb{R}, +, -, ., 0, 1, <)$, то будет ли эта функция локально вещественной частью голоморфной функции, которая определима в том же самом расширении? В предложении 2.1 доказано, что эта задача имеет положительное решение, если теорема Вейерштрасса о делении имеем место для системы колец определимых вещественно аналитических ростков в нуле пространства $\mathbb{R}^n$. Тот же ответ получается для o-минимального расширения вещественного поля, которое замкнуто относительно пфаффиана (предложение 2.6) для гармонических функций. В последнем параграфе показано, что теорема Вейерштрасса о делении не выполняется для колец ростков вещественных аналитических в $0 \in \mathbb{R}^n$ функций, которые определимы в o-минимальной структуре $(\overline{\mathbb{R}}, x^{\alpha_1}, \ldots, x^{\alpha_p})$, где $\alpha_1, \ldots, \alpha_p$ — вещественные иррациональные числа.

Ключевые слова: o-минимальные структуры, плюригармоническая функция, теорема Вейерштрасса о делении.

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