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TOPOLOGICAL UNIFIED (r, s)-ENTROPY OF CONTINUOUS MAPS ON QUASI-METRIC SPACES

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Abstract. The category of metric spaces is a subcategory of quasi-metric spaces. It is shown that the entropy of a map when symmetric properties is included is greater or equal to the entropy in the case that the symmetric property of the space is not considered. The topological entropy and Shannon entropy have similar properties such as nonnegativity, subadditivity and conditioning reduces entropy. In other words, topological entropy is supposed as the extension of classical entropy in dynamical systems. In the recent decade, different extensions of Shannon entropy have been introduced. One of them which generalizes many classical entropies is unified (r, s)-entropy. In this paper, we extend the notion of unified (r, s)-entropy for the continuous maps of a quasi-metric space via spanning and separated sets. Moreover, we survey unified (r, s)-entropy of a map for two metric spaces that are associated with a given quasi-metric space and compare unified (r, s)-entropy of a map of a given quasi-metric space and the maps of its associated metric spaces. Finally we define Tsallis topological entropy for the continuous map on a quasi-metric space via Bowen's definition and analyze some properties such as chain rule.

Key words: topological entropy, Tsallis entropy, Tsallis topological entropy, quasi-metric spaces.

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1. Introduction

In 1948 statistical entropy was introduced by C. E. Shannon as a basic concept in information theory, measuring the average missing information on a random source [1]. A. Renyi [2] and C. Tsallis [3] introduced one-parametric entropies, which include Shannon entropy in the special case. These entropies in a wide range of phenomena in diverse disciplines such as physics, chemistry, biology, medicine, economics, geophysics, statistical physics, abstract algebra, etc. are used.

One of the generalizations of Shannon's entropy involving two real-valued parameters is unified (r, s)-entropy which was introduced by P. N. Rathie and I. J. Tanjea [4]. A. E. Rastegin in 2011 studied some general properties of unified (r, s)-entropy [5]. Entropy measures have widely been adopted in studying features of quantum channels, A. E. Rastegin considered properties of quantum channels with the use of unified (r, s)-entropy [6].

Inspired by the notion of measure entropy in Ergodic Theory given by A. N. Kolmogorov [7] and Y. G. Sinai [8], in 1965 R. Adler, A. Konheim and M. H. McAndrew introduced the topological entropy of a continuous self-map of a compact space [9], and this concept was successively modified and generalized by R. Bowen [10], E. I. Dinaburg [11] and others. Topological entropy is an indicator of complicated (chaotic) behavior in dynamical systems. On the other hand, whether the topological entropy of a dynamical system is positive the system is chaotic.

The concept of topological unified (r,s)-entropy via Bowen's definition for continuous self-maps in dynamical systems on compact spaces by using separated and spanning sets was introduced in [12]. In the present paper we extend this notion and introduce unified (r,s)-entropy for the continuous maps of a quasi-metric space via spanning and separated sets.

We extend the notion of unified (r,s)-entropy for the continuous maps of a quasi-metric space via spanning and separated sets. Moreover, we survey unified (r,s)-entropy of a map for two metric spaces that are associated with a given quasi-metric space and we compare unified (r,s)-entropy of a map of a given quasi-metric space and the maps of its associated metric spaces.

Finally we define Tsallis topological entropy for the continuous map of a quasi-metric space via Bowen's definition and analyze some properties such as chain rule.

Paper organization: In Section 2 the important preliminaries and main concepts are stated. In Section 3 we extend the notion of unified (r,s)-entropy for the continuous maps of a quasi-metric space via spanning and separated sets. Moreover, we survey unified (r,s)-entropy of a map for two metric spaces that are associated with a given quasi-metric space and we compare unified (r,s)-entropy of a map of a given quasi-metric space and the maps of its associated metric spaces. In Section 4 the definition of Tsallis topological entropy for the continuous maps of a quasi-metric space via Bowen's definition similar to that of Kazemi et al. in [12] is presented and this definition to Tsallis joint topological entropy, Tsallis conditional topological entropy, and mutual Topological entropy is extended. Finally the relationship between conditional entropy and joint entropy in the chain rule of Tsallis topological entropy is studied.

2. Preliminaries

Firstly, we give the Stoltenberg's definition of quasi-metric spaces [13, 14]. Then we state the concept of topological entropy for the maps on quasi-metric spaces which are continuous according to a special topology defined by quasi-metrics according to the definition of sayyari et al. in [15].

Finally, we present the definitions of Shannon entropy, Renyi entropy, Tsallis entropy, the entropy of type r and unified (r,s)-entropy of a discrete random variables X.

Definition 2.1. Let X be a set, a quasi metric is defined as a function $e : X \times X \to [0,\infty)$ that satisfies the following axioms:

(1) $e(x,y) \geqslant 0$;

(2) $e(x,y) = 0 \Leftrightarrow x = y$;

(3) $e(x,z) \leqslant e(x,y) + e(y,z)$ for all $x,y,z \in X$.

(X,e) is called a quasi-metric space. So quasi-metric has all the properties of metrics except symmetry.

Now, we review and explain the salient conceptions and results of topological entropies of continuous self-maps on quasi-metric space. If (X,e) is a quasi-metric space, then the definition of open right t-ball centered at p is:

$$B_t^r(p) = \{x \in X : e(p,x) < t\},$$

$\{B_t^r(p),\, p \in X,\, t, r \in R\}$ is a base for a topology on X. Also, the definition of an open left t-ball centered at a point p is:

$$B_t^l(p) = \{x \in X :\ e(x,p) < t\},$$

$$B_t(p) = \{x \in X :\ e(x,p) < t,\, e(p,x) < t\} = B_t^l(p) \cap B_t^r(p).$$

If $T : X \to X$ is a continuous map and n is a natural number

$$e_n(x,y) = \max\left\{e\left(T^i(x), T^i(y)\right) :\ 0 \leqslant i \leqslant n-1\right\}$$

is a new quasi-metric e_n on X.

Definition 2.2. If $n \in \mathbb{N}$, $\epsilon > 0$ and K is a compact subset of X, then $F \subset X$ is called an (n,ϵ)-span of K with respect to T, if for any $x \in K$, there exists $y \in F$ such that $e_n(x,y) \leqslant \epsilon$ and $e_n(y,x) \leqslant \epsilon$.

If n is a natural number, $\epsilon > 0$ and K is a compact subset of X then the smallest cardinality of any (n,ϵ)-spanning set of K with respect to T is denoted by $r'_n(\epsilon, K)$. (If we are going to emphasise on T we can write $r'_n(\epsilon, K, T)$.) If K is a compact subset of X and $\epsilon > 0$, then

$$r'(\epsilon, K, T) = \limsup_{n\to\infty} \frac{\log r'_n(\epsilon, K)}{n}.$$

We also denote $r'(\epsilon, K, T)$ by $r'(\epsilon, K, T, e)$. The value of $r'(\epsilon, K)$ could be ∞ and $r'_n(\cdot, K)$ is a non-increasing map on $(0, +\infty)$ Let

$$h'(T,K) = \lim_{\epsilon\to 0} r'(\epsilon, K, T). \tag{1}$$

Then the topological entropy of T is

$$h'(T) = \sup_K h'(T,K),$$

where the supremum is taken over the collection of all compact subsets of X.

For a natural number n, $\epsilon > 0$ and a compact subset K of X, a subset E of X is called an (n,ϵ)-separated of K with respect to T, if it satisfies the following property: If $x, y \in E$ and $x \neq y$, then $e_n(x,y) > \epsilon$ or $e_n(y,x) > \epsilon$ (i. e., if $x, y \in E$ and $x \neq y$, then $y \in \cap_{i=0}^{n-1} T^{-i} B_\epsilon^r(T^i x)$ or $x \cap_{i=0}^{n-1} T^{-i} B_\epsilon^r(T^i y)$).

If n is a natural number, $\epsilon > 0$ and K is a compact subset of X then $s_n(\epsilon, K)$ denotes the largest cardinality of any (n,ϵ)-separated subset of K with respect to on T. (When we need to emphasize T we shall write $s'_n(\epsilon, K, T)$). If K is a compact subset of X, $\epsilon > 0$ then we define $s'(\epsilon, K, T)$ by

$$\limsup_{n\to\infty} \frac{\log s_n(\epsilon, K)}{n}.$$

We also write $s'_n(\epsilon, K, T, e)$ if we need to emphasis to the quasi-metric e. The value of $s'_n(\epsilon, K)$ could be infinity, and it is obvious that $s_n(\cdot, K)$ is a non-increasing map of $(0, +\infty)$.

$$h'(T,K) = \lim_{\epsilon\to 0} s'(\epsilon, K, T) \tag{2}$$

and

$$h'(T) = \sup_K h'(T,K), \tag{3}$$

where the supremum is taken over the collection of all compact subset of X.

We sometimes write $h'_e(T)$ instead of $h'(T)$ to emphasis the dependence on e.

In [15] it was proven that equations (1) and (2) are equal.

By the two following definitions, we can turn a quasi-metric space into a metric space.

DEFINITION 2.3. If (X,e) is a quasi-metric space then it is obvious that (X,d_e) is a metric space, where $d_e : X \times X \to [0,\infty)$ is a metric on X defined by:

$$d_e(x,y) = \frac{e(x,y)+e(y,x)}{2}. \tag{4}$$

If (X,e) is a quasi-metric space, then (X,d_e) is a metric space.

If $T : (X,d_e) \to (X,d_e)$ is a continuous map, then topological entropy is equal to:

$$h_{d_e}(T) = \sup_K \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log(r_n(\epsilon,K,T,d_e))}{n} = \sup_K \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log(s_n(\epsilon,K,T,d_e))}{n},$$

where the supremum is taken over the collection of all compact subsets of X and $r_n(\epsilon,K,T,d_e)$ denotes the smallest cardinality of any (n,ϵ)-spanning set for K with respect to the metric d_e and T, and $s_n(\epsilon,K,T,d_e)$ denotes the largest cardinality of any (n,ϵ)-separated subset of K with respect to the metric d_e and T. In [15] it was shown that $h_{d_e}(T) = h'_e(T)$.

Now we give another metric definition on (X,e).

DEFINITION 2.4. If (X,e) is a quasi-metric space, then (X,m_e) is a metric space, where $m_e(x,y) = \max\{e(x,y),e(y,x)\}$.

The topology generated by open sets $\{B_{m_e}(x,\epsilon), x\in X, \epsilon>0\}$ is the same as the topology generated by open sets $\{B_r(x), x\in X, r>0\}$. Therefor $h_{m_e}(T) = h'_e(T)$.

DEFINITION 2.5. Let X be a discrete random variable taking values in a finite set χ and $P_X(x)$ be a probability mass function. Then Shannon entropy is defined as:

$$H(X) = -\sum_{x\in\chi} P_X(x)\log P_X(x). \tag{5}$$

DEFINITION 2.6. Let X and Y be random variables taking values, respectively, in finite sets χ and γ with a joint distribution $p_{X,Y}(x,y)$. Shannon joint entropy formula is:

$$H(X,Y) = -\sum_{x\in\chi}\sum_{y\in\gamma} P_{X,Y}(x,y)\log P_{X,Y}(x,y). \tag{6}$$

The joint entropy is used to calculate the amount of uncertainty when two random variables X and Y happen together.

The formula $H(Y|X) = -\sum_{x\in\chi}\sum_{y\in\gamma} P_{X,Y}(x,y)\log P_{X,Y}(y|x)$ denotes conditional entropy of Y given X. If X and Y are independent, then $H(X,Y) = H(X)+H(Y)$.

The mutual information between two discrete random variables X,Y is given by

$$I(X:Y) = H(X)+H(Y)-H(X,Y).$$

So, if X and Y are independent, then $I(X:Y)$ is equal to zero and vice versa.

Rényi entropy or entropy of order r is defined by

$$H_r(X) = \frac{1}{1-r}\log\sum_{x\in\chi} P_X(x)^r \quad (\text{when } \ r>0,\ r\neq 1), \tag{7}$$

also, the definition of Tsallis entropy or entropy of degree r is:

$$H^r(X) = \frac{1}{1-r}\left(\sum_{x\in\chi} P_X(x)^r - 1\right) \quad (\text{when } \ r>0,\, r\neq 1) \tag{8}$$

or equivalently

$$H^r(X) = \sum_{x\in\chi} P_X(x)^r \ln_r P_X(x) \quad \left(\text{when } \ \ln_r(x) = \frac{x^{1-r}-1}{1-r}\right). \tag{9}$$

One of the basic properties of Tsallis entropy is the pseudoadditivity property for $r\neq 1$:

$$H^r(X\times Y) = H^r(X) + H^r(Y) + (1-r)H^r(X)H^r(Y). \tag{10}$$

The entropy of type r [16] is another extension of Shannon entropy, it was defined as:

$$ {}_rH(X) = \frac{1}{r-1}\left(\sum_{x\in\chi}\left(P_X(x)^{\frac{1}{r}}\right)^r - 1\right) \quad (\text{when } \ r>0,\, r\neq 1). \tag{11}$$

Tsallis entropy, Rényi entropy and entropy of type r coincide with Shannon entropy, when r tends to 1, so that they are extensions of Shannon entropy.

The following entropy involving two real parameters r and s is called (r,s)-entropy:

$$H_r^s(X) = \frac{1}{(1-r)s}\left[\left(\sum_{x\in\chi} P_X(x)^r\right)^s - 1\right] \quad (\text{when } \ r>0,\, r\neq 1,\, s\neq 0). \tag{12}$$

This entropy can be considered as an extension of the above four entropies. We can rewrite this entropy as follows:

$$H_r^s(X) = g_r^s(H_r(X)), \tag{13}$$

where

$$g_r^s(x) = \frac{1}{(1-r)s}\left(e^{(1-r)sx} - 1\right) \quad (\text{when } \ r>0,\, r\neq 1,\, s\neq 0). \tag{14}$$

It is easy to verify that g_r^s is an increasing function of x.

When $\alpha,\beta>0$ and $\alpha,\beta\neq 1$, by replacing $r=\alpha$ and $s=\frac{\beta-1}{\alpha-1}$, this entropy converts itself to Sharma and Mittal's entropy of order α and degree β [17].

By [4], we can write the above five entropies with respect to the parameters r and s in the following form:

$$E_r^s(X) = \begin{cases} H_r^s(X), & r\neq 1,\, s\neq 0,\\ H_r(X), & r\neq 1,\, s=0,\\ H^r(X), & r\neq 1,\, s=1,\\ {}_rH(X), & r\neq 1,\, s=\frac{1}{r},\\ H(X), & r=1. \end{cases} \tag{15}$$

3. Topological Unified (r,s)-Entropy on Quasi-Metric Spaces

Now, we extend the notion of unified (r,s)-entropy for the continuous maps of a quasi-metric space according to Bowen's definition. Moreover, we survey unified (r,s)-entropy of a map for two metric spaces that are associated with a given quasi-metric space and we compare unified (r,s)-entropy of a map of a given quasi-metric space and the maps of its associated metric spaces. Then we state the definition of joint topological unified (r,s)-entropy

DEFINITION 3.1. Suppose that, (X,e) is a quasi-metric space, $T: X \to X$ is a continuous transformation and $K \subseteq X$ is compact, when $s \neq 0$ and r is a positive number $r > 0$ and $r \neq 1$, then the definition of topological (r,s)-entropy for the continuous maps of a quasi-metric space is:

$$H^s_{e,r}(T,K) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{(1-r)s}\left(r'_n(T,K,\epsilon)^{\frac{1}{n}(1-r)s} - 1\right) \tag{16}$$

or

$$H^s_{e,r}(T,K) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{(1-r)s}\left(s'_n(T,K,\epsilon)^{\frac{1}{n}(1-r)s} - 1\right). \tag{17}$$

$$H^s_{e,r}(T) = \sup_K H^s_{e,r}(T,K), \tag{18}$$

where the supremum is taken over the collection of all compact subsets of X. According to equation (14), this formula can be rewritten in composite form with the topological entropy as follows:

$$H^s_{e,r}(T,K) = g^s_r(h'_e(T,K)) \tag{19}$$

and

$$H^s_{e,r}(T) = \sup_K H^s_{e,r}(T,K). \tag{20}$$

Sice g^s_r is an increasing function, we have

$$H^s_{e,r}(T) = g^s_r(h'_e(T)). \tag{21}$$

$$H^s_{e,r}(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{(1-r)s}\left(r'_n(T,\epsilon)^{\frac{1}{n}(1-r)s} - 1\right) \quad (\text{when } \ s \neq 0,\ r \neq 1,\ r > 0) \tag{22}$$

or

$$H^s_{e,r}(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{(1-r)s}\left(s'_n(T,\epsilon)^{\frac{1}{n}(1-r)s} - 1\right) \quad (\text{when } \ s \neq 0,\ r \neq 1,\ r > 0)\,. \tag{23}$$

When r approaches to 1, the limit of topological (r,s)-entropy coincides with topological entropy:

$$\lim_{r\to 1} H^s_{e,r}(T) = h'_e(T).$$

Similarly, when s approaches to 0, we have

$$\lim_{s\to 0} H^s_{d,r}(T) = h'_e(T).$$

According to definitions of m_e and d_e in Definition 2.3 and Definition 2.4, it is concluded that:

$$H^s_{e,r}(T) = g^s_r(h'_e(T)) = g^s_r(h_{d_e}(T)) = g^s_r(h_{m_e}(T)).$$

DEFINITION 3.2. Tsallis topological entropy or entropy of degree r and topological entropy of type r are relatively denoted by $H^r_e(T)$ and ${}_rH_e(T)$ and are defined as follows:

$${}_rH_e(T) = \frac{1}{r-1}\left(r'_n(T,\epsilon)^{\frac{r-1}{n}} - 1\right), \tag{24}$$

$$H^r_e(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{1-r}\left(r'_n(T,\epsilon)^{\frac{1-r}{n}} - 1\right). \tag{25}$$

We can write the above four topological entropies with respect to the parameters r, s in the following form:

$$E^s_{e,r}(T) = \begin{cases} H^s_{e,r}(T) & r \neq 1,\ s \neq 0, \\ h'_e(T) & r \neq 1,\ s = 0, \\ H^r_e(T) & r \neq 1,\ s = 1, \\ {}_rH_e(T) & r \neq 1,\ s = \frac{1}{r}, \\ h'_e(T) & r = 1, \end{cases}$$

$E^s_{e,r}(T)$ is called unified (r,s)-entropy.

We can see $H^{2-r}_e(T) = {}_rH_e(T)$, and when $r > 0$ we have $E^s_{e,r}(T) \geqslant 0$.

Lemma 3.1. *For $0 < r \leqslant 1$ and $r'_n(T,\epsilon) \neq 1$ we have $h'_e(T) \leqslant H^r_e(T)$.*

$\lhd$ Since for any $x > 0$, $\log x \leqslant x - 1$, we have:

$$h'_e(T) = \frac{1}{n}\log r'_n(T,\epsilon) = \frac{1}{1-r}\log r'_n(T,\epsilon)^{\frac{1}{n}(1-r)} \leqslant \frac{1}{1-r}\left(r'_n(T,\epsilon)^{\frac{1}{n}(1-r)} - 1\right) = H^r_e(T). \rhd$$

We state the definition of joint topological unified (r,s)-entropy.

Definition 3.3. Let (X,e) be a quasi-metric space and $T, T' \colon X \to X$, be two continuous transformations on X, the joint topological unified (r,s)-entropy is defined as:

$$E^s_{e,r}(T,T') = \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{(1-r)s}\left(r'_n(T,\epsilon)^{\frac{1}{n}(1-r)s} r'_n(T',\epsilon)^{\frac{1}{n}(1-r)s} - 1\right), \tag{26}$$

when $s \neq 0$, $r \neq 1$ and $r > 0$.

So

$$E^s_{e,r}(T,T') = E^s_{e,r}(T) + E^s_{e,r}(T') + (1-r)sE^s_{e,r}(T)E^s_{e,r}(T').$$

When $r > 1$, it is concluded that:

$$E^s_{e,r}(T,T') \leqslant E^s_{e,r}(T) + E^s_{e,r}(T'),$$

i.e., in this case, topological unified (r,s)-entropy satisfies in subadditivity property.

Also, $\max\{E^s_r(T), E^s_r(T')\} \leqslant E^s_r(T,T')$, i.e., joint topological unified (r,s)-entropy is greater than two corresponding topological unified (r,s)-entropy.

Monotonicity of the topological unified (r,s)-entropy was studied in [12]. Now, we extend this subject for a quasi-metric space.

Theorem 3.1. *Let (X,e) be a quasi-metric space and $T : X \to X$, be a continuous transformation on X. Then we have*

1) *For any $s > 0$, $E^s_{e,r}(T)$ is decreasing with respect to $r \in (0,1) \cup (1,\infty)$.*

2) *If $0 < r < 1$, then $E^s_{e,r}(T)$ is increasing with respect to $s \in (-\infty,\infty)$, and if $r > 1$, then $E^s_r(T)$ is decreasing with respect to $s \in (-\infty,\infty)$.*

3) *For any $r > 0$, $E^s_{e,r}(T)$ is convex with respect to $s \in (-\infty,+\infty)\backslash\{0\}$.*

For any $s > 0$, $E^s_{e,r}(T)$ is decreasing with respect to $r \in (0,1) \cup (1,\infty)$.

4. Tsallis Topological Entropy

In this section we define Tsallis topological entropy for the continuous map of a quasi-metric space via Bowen's definition.

Definition 4.1. Suppose that, (X, e) is a compact quasi-metric space, $T : X \to X$ is a continuous transformation, when r is a positive number, then the definition of Tsallis topological entropy for the continuous map of a quasi-metric space is:

$$H^r(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{1-r}\left(r'_n(T,\epsilon)^{\frac{1-r}{n}} - 1\right) = \frac{1}{1-r}\left(e^{(1-r)h_e(T)} - 1\right) \tag{27}$$

or equivalently

$$H^r(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} -r'_n(T,\epsilon)\ln_r r'_n(T,\epsilon)^{\frac{-1}{n}}$$

and

$$H^r(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} -\frac{1}{1-r}\left(s'_n(T,\epsilon)^{\frac{1-r}{n}} - 1\right) = \frac{1}{1-r}\left(e^{(1-r)h_e(T)} - 1\right) \tag{28}$$

or equivalently

$$H^r(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} -s'_n(T,\epsilon)^{\frac{1-r}{n}}\ln_r s'_n(T,\epsilon)^{\frac{-1}{n}}.$$

It is obvious that equations (27) and (28) are existing and equal.

As r tends to 1, the limit of Tsallis topological entropy coincides with topological entropy, since Tsallis topological entropy is an extension of topological entropy. $H^r(T)$ is nonnegative and decreasing with respect to r. Also $H^r_e(T) = 0$ if and only if $h'_e(T) = 0$.

Now, we define Tsallis joint and conditional entropy for two continuous transformations on compact quasi-metric space (X, e).

Definition 4.2. If (X, e) is a compact quasi-metric space and $T, T' : X \to X$ are two continuous transformations on X, the definition of Tsallis joint topological entropy for $r > 0$ and $r \neq 1$ is:

$$H^r_e(T, T') = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{1-r}\left(r'_n(T,\epsilon)^{\frac{1-r}{n}} r'_n(T',\epsilon)^{\frac{1-r}{n}} - 1\right). \tag{29}$$

Furthermore, we define Tsallis joint topological entropy for m continuous transformations $T_1, \ldots, T_m$ on compact quasi-metric space (X, e), when $r > 0$ and $r \neq 1$ as:

$$H^r_e(T_1, T_2, \ldots, T_m) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{1-r}\left(r'_n(T_1,\epsilon)^{\frac{1-r}{n}} r'_n(T_2,\epsilon)^{\frac{1-r}{n}} \ldots r'_n(T_m,\epsilon)^{\frac{1-r}{n}} - 1\right). \tag{30}$$

Proposition 4.1. *If (X, e) is a compact quasi-metric space and $T : X \to X$ and $T' : X \to X$ are two continuous transformations on X, then to fulfill the above assumptions, we must have*

$$H^r_e(T, T') = H^r_e(T) + H^r_e(T') + (1-r)H^r_e(T)H^r_e(T'). \tag{31}$$

$\lhd$ It is clear that:

$$H^r_e(T, T') = \frac{1}{1-r}\left(r'_n(T,\epsilon)^{\frac{1-r}{n}} r'_n(T',\epsilon)^{\frac{1-r}{n}} - 1\right)$$

$$= \frac{1}{1-r}\left(r'_n(T,\epsilon)^{\frac{1-r}{n}} - 1 + r'_n(T',\epsilon)^{\frac{1-r}{n}} - 1 + \left(r'_n(T,\epsilon)^{\frac{1-r}{n}} - 1\right)\left(r'_n(T',\epsilon)^{\frac{1-r}{n}} - 1\right)\right)$$

$$= H^r_e(T) + H^r_e(T') + (1-r)H^r_e(T)H^r(T'). \rhd$$

For $r \geqslant 1$, due to equation (31), the subadditivity inequality holds:

$$H^r_e(T, T') \leqslant H^r_e(T) + H^r_e(T'), \tag{32}$$

and for $0 \leqslant r < 1$ the subadditivity inequality does not hold.

Corollary 4.1. *If (X, e) is a compact quasi-metric space and $T_1, \dots, T_n : X \to X$ are continuous transformations on X, then for $r \geqslant 1$ is concluded that:*

$$H_e^r(T_1, T_2, \dots, T_n) \leqslant \sum_{i=1}^{n} H_e^r(T_i).$$

Now, we define Tsallis conditional topological entropy.

Definition 4.3. If (X, e) is a compact metric space and $T : X \to X$ and $T' : X \to X$ are two continuous transformations on X, we define Tsallis conditional topological entropy by the following formula:

$$H_e^r(T|T') = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{1-r}\Big(r_n'(T',\epsilon)^{\frac{1-r}{n}}(r_n'(T,\epsilon)^{\frac{1-r}{n}} - 1)\Big) \quad (\text{when } \; r > 0,\, r \neq 1). \tag{33}$$

Therefore the strong subadditivity is verified for $r \geqslant 1$.

Proposition 4.2. *If (X, e) is a quasi-metric space and $T, T' : X \to X$ are continuous transformations on X, then*
(1) $H_e^r(T|T') = H_e^r(T, T') - H_e^r(T')$;
(2) $H_e^r(T|T') \leqslant H_e^r(T, T')$;
(3) $H_e^r(T) \leqslant H_e^r(T, T')$.

$\lhd$ (1) and (2) are clear.

For (3), let $r > 1$. Then

$$r_n'(T,\epsilon)^{\frac{1-r}{n}} r_n'(T',\epsilon)^{\frac{1-r}{n}} \leqslant r_n'(T,\epsilon)^{\frac{1-r}{n}}.$$

Therefore

$$\frac{1}{1-r}\Big(r_n'(T,\epsilon)^{\frac{1-r}{n}} r_n'(T',\epsilon)^{\frac{1-r}{n}}\Big) \geqslant \frac{1}{1-r} r_n'(T,\epsilon)^{\frac{1-r}{n}}$$

and the proof is concluded. The proof for $0 < r < 1$ is similar. $\rhd$

We tend to state the chain rule for Tsallis topological entropy.

Lemma 4.1. *We have the following equations:*
(1) $H_e^r(T_1, T_2, T_3) = H_e^r(T_1, T_2|T_3) + H_e^r(T_3)$;
(2) $H_e^r(T_1, T_2|T_3) = H_e^r(T_1|T_3) + H_e^r(T_2|T_1, T_3)$.

$\lhd$ From Definition 4.2 and equation (33), we conclude

$$H_e^r(T_1, T_2|T_3) = H_e^r(T_1, T_2, T_3) - H_e^r(T_3),$$

$$H_e^r(T_2|T_1, T_3) = H_e^r(T_1, T_2, T_3) - H_e^r(T_1, T_3)$$

and

$$H_e^r(T_1, T_2) = H_e^r(T_1) + H_e^r(T_2|T_1).$$

Therefore

$$\begin{aligned} H_e^r(T_2|T_1, T_3) = H_e^r(T_1, T_2, T_3) - H_e^r(T_1, T_3) = H_e^r(T_1, T_2|T_3) + H_e^r(T_3) \\ - (H_e^r(T_1|T_3) + H_e^r(T_3)) = H_e^r(T_1, T_2|T_3) - H_e^r(T_1|T_3). \end{aligned}$$

From part (2) of proposition 4.2, we conclude that $H_e^r(T_1|T_3) \leqslant H_e^r(T_1, T_2|T_3)$. $\rhd$

In the following theorem, the chain rule is proven.

Theorem 4.1. *Assume that $T_1, T_2, \ldots, T_m : X \to X$ are continuous transformations on compact quasi-metric space (X, e), we have*

$$H_e^r(T_1, T_2, \ldots, T_m) = \sum_{i=1}^{m} H_e^r(T_i|T_{i-1}, \ldots, T_1). \tag{34}$$

So by exploiting the chain rule (34), we have

$$H_e^r(T_1, T_2, \ldots, T_m|T_{m+1}) = \sum_{i=1}^{m} H_e^r(T_i|T_{i-1}, \ldots, T_1, T_{m+1}). \tag{35}$$

In the next theorem, the relationship between conditional and joint topological Tsallis entropy is studied.

Theorem 4.2. *If the conditions of Theorem 4.1 is fulfilled, then:*

$$H_e^r(T_1, T_2, \ldots, T_m) = \frac{1}{1-r}\left[\prod_{i=1}^{m} H_e^r(T_i|T_1, \ldots, T_{i-1}) - 1\right].$$

Theorem 4.3. *For $r \geqslant 1$, the strong subadditivity satisfies, i.e.,*

$$H_e^r(T_1, T_2, T_3) + H_e^r(T_3) \leqslant H_e^r(T_1, T_3) + H_e^r(T_2, T_3). \tag{36}$$

◁ First, we show that

$$H_e^r(T_1|T_2, T_3) \leqslant H_e^r(T_1|T_3). \tag{37}$$

$$\begin{aligned} &H_e^r(T_1|T_2, T_3) \leqslant H_e^r(T_1|T_3) \\ \Longleftrightarrow &\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{1-r}\left(r'_n(T_1,\epsilon)^{\frac{1-r}{n}} r'_n(T_2,\epsilon)^{\frac{1-r}{n}} r_n(T_3,\epsilon)^{\frac{1-r}{n}} - r'_n(T_2,\epsilon)^{\frac{1-r}{n}} r'_n(T_3,\epsilon)^{\frac{1-r}{n}}\right) \\ &\leqslant \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{1-r}\left(r'_n(T_1,\epsilon)^{\frac{1-r}{n}} r'_n(T_3,\epsilon)^{\frac{1-r}{n}} - r'_n(T_3,\epsilon)^{\frac{1-r}{n}}\right) \\ \Longleftrightarrow &\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{1-r}\left(r'_n(T_1,\epsilon)^{\frac{1-r}{n}} r'_n(T_3,\epsilon)^{\frac{1-r}{n}} (r'_n(T_2,\epsilon)^{\frac{1-r}{n}} - 1)\right) \\ &\leqslant \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{1-r} r'_n(T_3,\epsilon)^{\frac{1-r}{n}}\left(r'_n(T_2,\epsilon)^{\frac{1-r}{n}} - 1\right). \end{aligned}$$

Since for $r > 1$, $\frac{1}{1-r}(r'_n(T_2,\epsilon)^{\frac{1-r}{n}} - 1)$ is nonnegative. Therefore

$$\begin{gathered} H_e^r(T_1|T_2, T_3) \leqslant H_e^r(T_1|T_3) \Longrightarrow \\ H_e^r(T_1, T_2, T_3) - H_e^r(T_2, T_3) \leqslant H_e^r(T_1, T_3) - H_e^r(T_3) \Longrightarrow \\ H_e^r(T_1, T_2, T_3) + H_e^r(T_3) \leqslant H_e^r(T_1, T_3) + H_e^r(T_2, T_3). \end{gathered} \tag{38}$$

For $r = 1$, (38) coincides with classical (Shannon) case. ▷

Proposition 4.3. *For $r > 1$, we have*

$$H_e^r(T_m|T_1) \leqslant H_e^r(T_2|T_1) + \ldots + H_e^r(T_m|T_{m-1}).$$

◁ From equations (37) and (35), we conclude

$$H_e^r(T_1, \ldots, T_m) \leqslant H_e^r(T_1) + H_e^r(T_2|T_1) + \ldots + H_e^r(T_m|T_{m-1}). \tag{39}$$

Therefore, from (39) and the definition of conditional topological Tsallis entropy is obtained that:

$$H_e^r(T_m|T_1) \leqslant H_e^r(T_2,\dots,T_m|T_1) \leqslant H_e^r(T_1,\dots,T_m) - H_e^r(T_1) \\ \leqslant H_e^r(T_2|T_1) + \ldots + H_e^r(T_m|T_{m-1}). \ \triangleright$$

5. Conclusion

In this paper, the notion of unified (r, s)-entropy for the continuous maps of a quasi-metric space via spanning and separated sets is defined. The concept of topological entropy for maps on quasi-metric spaces which are continuous according to the sayyari's definition is stated. We will generalize the leibler-Kullback distance for two continuous maps on metric and compact and quasi-metric space.

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ТОПОЛОГИЧЕСКАЯ УНИФИЦИРОВАННАЯ (r, s)-ЭНТРОПИЯ НЕПРЕРЫВНЫХ ОТОБРАЖЕНИЙ В КВАЗИМЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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Аннотация. Категория метрических пространств является подкатегорией квазиметрических пространств. Показано, что энтропия отображения в пространстве с условиями симметричности больше или равна энтропии того случая, когда условия симметричности не предполагаются. Топологическая энтропия и энтропия Шеннона имеют схожие свойства такие, как неотрицательность, субаддитивность и снижение условной энтропии. Другими словами, топологическая энтропия рассматривается как расширение классической энтропии в динамических системах. В последнее десятилетие были введены различные обобщения энтропии Шеннона. Одной из них, обобщающей многие классические виды энтропии, является унифицированная (r, s)-энтропия. В данной работе понятие унифицированной (r, s)-энтропии распространяется на непрерывные отображения в квазиметрических пространствах посредством связующих и разделяющих множеств. Далее, рассматривается унифицирующая (r, s)-энтропия отображения в двух метрических пространствах, ассоциированных с квазиметрическим пространством и сравниваются унифицированные (r, s)-энтропии отображения в данном квазиметрическом пространстве и в ассоциированных метрических пространствах. Наконец, определяется топологическая энтропия Цаллиса для непрерывных отображений в квазиметрических пространствах посредствм определения Бовена и изучаются некоторые свойства, такие как цепное правило.

Ключевые слова: энтропия Цаллиса, топологическая энтропия Цаллиса, квазиметрическое пространство.

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