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UNICITY ON ENTIRE FUNCTIONS CONCERNING THEIR DIFFERENCE OPERATORS AND DERIVATIVES

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Abstract. In this paper we study the uniqueness of entire functions concerning their difference operator and derivatives. The idea of entire and meromorphic functions relies heavily on this direction. Rubel and Yang considered the uniqueness of entire function and its derivative and proved that if $f(z)$ and $f'(z)$ share two values a, b counting multilicities then $f(z) \equiv f'(z)$. Later, Li Ping and Yang improved the result given by Rubel and Yang and proved that if $f(z)$ is a non-constant entire function and a, b are two finite distinct complex values and if $f(z)$ and $f^{(k)}(z)$ share a counting multiplicities and b ignoring multiplicities then $f(z) \equiv f^{(k)}(z)$. In recent years, the value distribution of meromorphic functions of finite order with respect to difference analogue has become a subject of interest. By replacing finite distinct complex values by polynomials, we prove the following result: Let $\Delta f(z)$ be trancendental entire functions of finite order, $k \geqslant 0$ be integer and P_1 and P_2 be two polynomials. If $\Delta f(z)$ and $f^{(k)}$ share P_1 CM and share P_2 IM, then $\Delta f \equiv f^{(k)}$. A non-trivial proof of this result uses Nevanlinna's value distribution theory.

Key words: difference operator, shared values, finite order, uniqueness, entire function, polynomials.

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1. Introduction and Main Results

The reader is presumed to be familiar with the fundamental notations and conclusions of Nevanlinna's value distribution theory of meromorphic functions [1, 2]. $S(r, f)$ means that $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithamic measure, and

$$\lambda(f) = \lim_{r\to\infty} \sup \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r}, \quad \delta(P_1, f) = \lim_{r\to\infty} \inf \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \lim_{r\to\infty} \sup \frac{N\left(r, \frac{1}{f}\right)}{T(r, f)}$$

stand for the exponents of convergence of zero sequence of f and the deficiency of f at the point a, respectively. For a nonconstant meromorphic function h, we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o(T(r, h))$, as r runs to infinity outside of a set $E \subset (0, +\infty)$ of finite linear measure. We say that h is a small function of f, if $T(r,h) = S(r,f)$. In the sequel, we denote by I a set of infinite linear measure not necessarily the same in all its occurrences.

We say that f and g share the value a IM (ignoring multiplicities), if f and g have the same a point. If f and g have the same a point with the same multiplicities, then we say f, g share the value a CM (counting multiplicities).

Definition 1 [3]. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a points of $f(z)$ where an a point of mulplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f, g share the value a with weight k.

We write f and g share (a, k) to mean that f, g share the value a with weight k.

Rubel and Yang Chung-Chun [4] considered the uniqueness of an entire function and its derivative. They proved the following.

Theorem 1. *Let $f(z)$ be a non-constant entire function, let a, b be two finite distinct values. If $f(z)$ and $f'(z)$ share a, b CM, then $f(z) \equiv f'(z)$.*

Li Ping and Yang Chung-Chun [5] improved Theorem 1 and proved.

Theorem 2. *Let $f(z)$ be a non-constant entire function, and let a, b be two finite distinct complex values. If $f(z)$ and $f^{(k)}(z)$ share a CM, and share b IM. Then $f(z) \equiv f^{(k)}(z)$.*

The value distribution of meromorphic functions of finite order with respect to difference analogue has become a subject of some interests, see [6–16].

Theorem 3 [17]**.** *Suppose that $f(z)$ and $g(z)$ are nonconstact meromorphic functions. If f, g share 0, 1, ∞ CM and $\overline{N}\big(r, \frac{1}{f}\big) + \overline{N}(r, f) < (d + o(1))T(r, f)$ for $r \in I$ and $r \in \infty$, where d is a positive number satisfying $0 < d < \frac{1}{2}$, which $I \subset (0, +\infty)$ is a subset of infinite linear measure, then $f = g$ or $f.g = 1$.*

Theorem 4 [4]**.** *Let f be a nonconstant entire function. If f shares two distinct finite values CM with f', then $f \equiv f'$.*

More results on uniqueness of f' with its n-th derivative $f^{(n)}$ were obtained by several authors (see [18–20]). In view of the progress on the difference analogues of classical Nevanlinna theory of meromorphic functions [21, 22], it is quite natural to investigate the uniqueness problems of meromorphic functions and their difference operators (see [23–26]).

Example 1. Let $f(z) = e^{Az}$, where $A \neq 0$ is a constant. Then $f^{(k)} = A^k e^{Az}$ and $\Delta f = f(z+1) - f(z) = (e^A - 1)e^{Az}$. Clearly, $\Delta(f)$ and $f^{(k)}$ share 0 CM and ∞ IM, and that $\mu = 1$. We can choose A such that $e^A - 1 \neq A^k$, and so $f' \not\equiv \Delta(f)$.

Theorem 5 [27]**.** *Let $f(z)$ be a trancendental entite function of finite order, let $\eta \neq 0$ be a finite complex number, $n \geqslant 1$, $k \geqslant 0$ be two integers and let a, b be two distinct finite complex values. If $f(z)$ and $\big(\Delta_\eta^n f(z)\big)^{(k)}$ share a CM and share b IM, then $f(z) \equiv \big(\Delta_\eta^n f(z)\big)^{(k)}$.*

Lemma 1 [9]**.** *Let Δf be a nonconstant meromorphic function of finite order, let $\eta \neq 0$ be a finite complex number. Then*

$$m\left(r, \frac{\Delta f(z+\eta)}{\Delta f}\right) = S(r, \Delta f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2 [28, Lemma 4.3]**.** *Let Δf be a nonconstant meromorphic function. Suppose that the polynomials P_j, $j = 0, 1, \ldots, q$, $q > p$, and let $P(\Delta f) = a_0(\Delta f)^p + a_1(\Delta f)^{p-1} + \ldots + a_p$ $(a_0 \neq 0)$ is a polynomial of degree p with constant coefficient a_j, $j = 0, 1, \ldots, p$. Then*

$$m\left(r, \frac{P(\Delta f)(\Delta f)'}{(\Delta f - P_1)(\Delta f - P_2)\ldots(\Delta f - P_q)}\right) = S(r, \Delta f).$$

Lemma 3. *Let Δf and Δg be two non constant entire functions, and let P_1, P_2 be two polynomials. If*

$$H=\frac{\Delta f'}{(\Delta f-P_1)(\Delta f-P_2)}-\frac{\Delta g'}{(\Delta g-P_1)(\Delta g-P_2)}\equiv 0,$$

and Δf and Δg share P_1 CM, and share P_2 IM, then either

$$2T(r,\Delta f)\leqslant \overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f)$$

or

$$\Delta f\equiv\Delta g.$$

◁ Integrating H which leads to

$$\frac{\Delta g-P_2}{\Delta g-P_1}=C\frac{\Delta f-P_2}{\Delta f-P_1},$$

where C is a nonzero constant.

If $C=1$, then $\Delta f\equiv\Delta g$. If $C\neq 1$, then from above, we have

$$\frac{P_1-P_2}{\Delta g-P_1}\equiv\frac{(C-1)\Delta f-CP_2+P_1}{\Delta f-P_1}$$

and

$$T(r,\Delta f)=T(r,\Delta g)+S(r,\Delta f)+S(r,\Delta g).$$

Obviously, $\frac{CP_1-P_2}{C-1}\neq a$ and $\frac{CP_1-P_2}{C-1}\neq b$. It follows that $N\left(r,\frac{1}{\Delta f-\frac{CP_1-P_2}{C-1}}\right)=0$. Then by the Second Fundamental Theorem,

$$\begin{aligned}2T(r,\Delta f)&=\overline{N}(r,\Delta f)+\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+\overline{N}\left(r,\frac{1}{\Delta f-\frac{CP_1-P_2}{C-1}}\right)\\&+S(r,\Delta f)\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f),\end{aligned}$$

that is

$$2T(r,\Delta f)\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f).\ ▷$$

Lemma 4. *Let Δf be a transcendental entire function of finite order, k be positive integer, let P_1 be a nonzero complex value or constant. If Δf and $f^{(k)}$ share P_1 CM, and $N\big(r,\frac{1}{f^{(k)}}\big)=S(r,\Delta f)$, then one of the following cases must occur:*

- $f^{(k)}=He^p$, *where p is a polynomial, and $H\not\equiv 0$ is a small function of e^p.*
- $T(r,e^p)=S(r,\Delta f)$.

◁ Since Δf is a transcendental entire function of finite order, Δf and $f^{(k)}$ share P_1 CM, then there is a polynomial p such that

$$\Delta f-P_1=e^p(f^{(k)})-P_1e^p. \tag{1}$$

Set $g=f^{(k)}$. It follows by (1) that

$$g=(ge^p)^{(k)}-(P_1e^p)^{(k)}. \tag{2}$$

Then we rewrite (2) as

$$1+\frac{(P_1e^p)^{(k)}}{g}=De^p, \tag{3}$$

where

$$D=\frac{(ge^p)^{(k)}}{ge^p}. \tag{4}$$

Note that $N\left(r,\frac{1}{(f)^{(k)}}\right)=N\left(r,\frac{1}{g}\right)=S(r,f)$, then by Lemma 1 we have

$$T(r,D)=T\left(r,\frac{(ge^p)^{(k)}}{\Delta ge^p}\right).$$

Next we discuss two cases.

Case 1: $e^{-p}-D\not\equiv 0$. Rewrite (3) as

$$ge^p(e^{-p}-D)=(P_1e^p)^{(k)}. \tag{5}$$

When $D\equiv 0$, (5) implies

$$g=He^p. \tag{6}$$

Here $H\not\equiv 0$ is a small function of e^p.
When $D\not\equiv 0$, it follows from (5) that $N\left(r,\frac{1}{e^{-p}-D}\right)=S(r,f)$. Then use the Second Fundamental Theorem to e^p we can obtain

$$T(r,e^p)=T(r,e^{-p})+O(1)\leqslant\overline{N}(r,e^p)+\overline{N}\left(r,\frac{1}{e^{-p}}\right)+\overline{N}\left(r,\frac{1}{e^{-p}-D}\right)+O(1)=S(r,\Delta f).$$

Case 2: $e^{-p}-D\equiv 0$. It implies that $T(r,e^p)=T(r,e^{-p})+O(1)=S(r,\Delta f)$, a contradiction. From above discussions, we get $T(r,e^p)=S(r,\Delta f)$. ▷

Theorem 6. *Let $\Delta f(z)$ be a transcendental entire functions of finite order, k be integer such that $k\geqslant 0$ and let P_1 and P_2 be two polynomials. If $\Delta f(z)$ and $f^{(k)}$ share P_1 CM and share P_2 IM, then $\Delta f\equiv f^{(k)}$.*

◁ If $\Delta f\equiv f^{(k)}$, there is nothing to prove. Solve $\Delta f\not\equiv f^{(k)}$. Since Δf is a transcenedental entire function of finite order, Δf and $f^{(k)}$ share P_1 CM, then we get

$$\frac{f^{(k)-}P_1}{\Delta f-P_1}=e^Q, \tag{7}$$

where Q is a polynomial.

Since Δf and $f^{(k)}$ share P_1 CM and share P_2 IM, then by second fundamental theorem and Lemma 1 we have

$$\begin{aligned}T(r,\Delta f)&\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f)\\ &\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-P_2}\right)\leqslant N\left(r,\frac{1}{\Delta f-f^{(k)}}\right)+S(r,\Delta f)\\ &\leqslant T\big(r,\Delta f-f^{(k)}\big)+S(r,\Delta f)\leqslant m\big(r,\Delta f-f^{(k)}\big)+S(r,\Delta f)\\ &\leqslant m(r,\Delta f)+m\left(r,1-\frac{f^{(k)}}{\Delta f}\right)+S(r,\Delta f)\leqslant T(r,f)+S(r,f),\end{aligned}$$

$$T(r,\Delta f)=\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(\Delta f-P_2\right)+S(r,\Delta f). \tag{8}$$

According to (7) and (8) we have

$$T(r,\Delta f)=T\big(r,\Delta f-f^{(k)}\big)+S(r,\Delta f)\leqslant N\left(r,\frac{1}{\Delta f-f^{(k)}}\right)+S(r,\Delta f) \tag{9}$$

and

$$T\big(r,e^{Q}\big)=m\big(r,e^{Q}\big)=m\left(r,\frac{f^{(k)}-P_1}{\Delta f-P_1}\right)\leqslant m\left(r,\frac{1}{\Delta f-P_1}\right)+S(r,\Delta f). \tag{10}$$

Then it follows from (7) and (9) that

$$\begin{aligned} m\left(r,\frac{1}{\Delta f-P_1}\right)&=m\left(r,\frac{e^{Q}-1}{\Delta f-f^{(k)}}\right)\leqslant m\left(r,\frac{1}{\Delta f-f^{(k)}}\right)+m\big(r,e^{Q}-1\big)\\ &\leqslant T\big(r,e^{Q}\big)+S(r,\Delta f). \end{aligned} \tag{11}$$

Then by (10) and (11)

$$T\big(r,e^{Q}\big)=m\left(r,\frac{1}{\Delta f-P_1}\right)+S(r,\Delta f). \tag{12}$$

On the other hand, (1) can be rewritten as

$$\frac{f^{(k)}-\Delta f}{\Delta f-P_1}=e^{Q}-1, \tag{13}$$

which implies

$$\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)\leqslant\overline{N}\left(r,\frac{1}{e^{Q}-1}\right)=T\big(r,e^{Q}\big)+S(r,\Delta f). \tag{14}$$

Then by (8), (12) and (14)

$$\begin{aligned} m\left(r,\frac{1}{\Delta f-P_1}\right)+N\left(r,\frac{1}{\Delta f-P_1}\right)&=\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f)\\ &\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{e^{Q}-1}\right)+S(r,\Delta f)\\ &\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+m\left(r,\frac{1}{\Delta f-P_1}\right)+S(r,\Delta f), \end{aligned}$$

$$N\left(r,\frac{1}{\Delta f-P_1}\right)=\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+S(r,\Delta f), \tag{15}$$

and then

$$\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)=T\big(r,e^{Q}\big)+S(r,\Delta f). \tag{16}$$

Set

$$\phi=\frac{(\Delta f)'\big(\Delta f-f^{(k)}\big)}{(\Delta f-P_1)(\Delta f-P_2)} \tag{17}$$

and
$$\psi=\frac{f^{(k+1)}\big(\Delta f-f^{(k)}\big)}{\big(f^{(k)}-P_1\big)\big(f^{(k)}-P_2\big)}. \tag{18}$$
Easy to know that ϕ is an entire function by Lemma 1 and Lemma 2 we have
$$\begin{aligned}T(r,\phi)=m(r,\phi)&=m\left(r,\frac{(\Delta f)'\big(\Delta f-f^{(k)}\big)}{(\Delta f-P_1)(\Delta f-P_2)}\right)+S(r,\Delta f)\\ &\leqslant m\left(r,\frac{(\Delta f)'(\Delta f)}{(\Delta f-P_1)(\Delta f-P_2)}\right)m\left(r,1-\frac{f^{(k)}}{\Delta f}\right)+S(r,\Delta f)=S(r,\Delta f),\end{aligned}$$
that is
$$T(r,\phi)=S(r,\Delta f). \tag{19}$$
Obiously Let $d=P_1-k(P_1-P_2)$, $k\neq 0$, by Lemmas 1 and 2, we obtain
$$\begin{aligned}&m\left(r,\frac{1}{\Delta f}\right)+m\left(r,\frac{1}{(P_2-P_1)\phi}\left(\frac{(\Delta f)'}{\Delta f-P_1}-\frac{(\Delta f)'}{\Delta f-P_2}\right)\left(\frac{1-(\Delta f)'}{\Delta f}\right)\right)\\ &\leqslant m\left(r,\frac{1}{\phi}\right)+m\left(r,\left(\frac{(\Delta f)'}{\Delta f-P_1}-\frac{(\Delta f)'}{\Delta f-P_2}\right)\right)+m\left(r,\frac{(\Delta f)'}{\Delta f}\right)+S(r,\Delta f)=S(r,\Delta)\end{aligned} \tag{20}$$
and
$$\begin{aligned}m\left(r,\frac{1}{\Delta f-d}\right)&=m\left(r,\frac{(\Delta f)'\big(\Delta f-f^{(k)}\big)}{(\Delta f-P_1)(\Delta f-P_2)(\Delta f-d)}\right)\\ &\leqslant m\left(r,\frac{1-f^{(k)}}{\Delta f}\right)+m\left(r,\frac{(\Delta f)'\big(\Delta f-f^{(k)}\big)}{(\Delta f-P_1)(\Delta f-P_2)(\Delta f-d)}\right)+S(r,\Delta f)=S(r,\Delta f).\end{aligned} \tag{21}$$
Set
$$\phi=\frac{(f)^{(k+1)}}{\big(f^{(k)}-P_1\big)\big(f^{(k)}-P_2\big)}-\frac{(\Delta f)'}{(\Delta f-P_1)(\Delta f-P_2)}. \tag{22}$$
We discuss two cases

Case 1: $\phi\equiv 0$. Integrating both side of (22) which leads to
$$\frac{\Delta f-P_2}{\Delta f-P_1}=c\frac{f^{(k)}-P_2}{f^{(k)}-P_1}, \tag{23}$$
where c is a non zero constant. Then by Lemma 3 we see that
$$2T(r,\Delta f)\leqslant\overline{N}\left(r,\frac{1}{\Delta f-P_1}\right)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f), \tag{24}$$
which contradicts with (8).

Case 2: $\phi\not\equiv 0$. By (9), (19) and (22) we can obtain
$$\begin{aligned}m(r,\Delta f)&=m\big(r,\Delta f-f^{(k)}\big)+S(r,\Delta f)=m\left(r,\frac{\phi\big(\Delta f-f^{(k)}\big)}{\phi}\right)+S(r,\Delta f)\\ &=m\left(r,\frac{\psi-\phi}{\phi}\right)+S(r,\Delta f)\leqslant T\left(r,\frac{\phi}{\psi-\phi}\right)+S(r,\Delta f)\\ &\leqslant T(r,\psi-\phi)+T(r,\phi)+S(r,\Delta f)\leqslant T(r,\psi)+\overline{N}\left(r,\frac{1}{\Delta f-P_2}\right)+S(r,\Delta f).\end{aligned} \tag{25}$$

On the other hand

$$\begin{aligned} T(r,\psi) &= T\left(r, \frac{f^{(k+1)}\left(\Delta f - f^{(k)}\right)}{\left(f^{(k)}-P_1\right)\left(f^{(k)}-P^{(2)}\right)}\right) \\ &= m\left(r, \frac{f^{(k+1)}\left(\Delta f - f^{(k)}\right)}{\left(f^{(k)}-P_1\right)\left(f^{(k)}-P^{(2)}\right)}\right) + S(r,\Delta f) \\ &\leqslant m\left(r, \frac{f^{(k+1)}}{f^{(k)}-P_2}\right) + m\left(r, \frac{\Delta f - f^{(k)}}{f^{(k)}-P_1}\right) \\ &\leqslant m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r,\Delta f) = \overline{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r,\Delta f). \end{aligned} \tag{26}$$

Hence combining (25) and (26) we obtain

$$T(r,\Delta f) \leqslant 2\overline{N}\left(r, \frac{1}{\Delta f - P_2}\right) + S(r,\Delta f). \tag{27}$$

Next, case 2 is divided into two subcases.

Subcase 2.1. $P_1 = 0$. Then by (7) and Lemma 1 we get

$$m\left(r, e^{Q}\right) = m\left(r, \frac{f^{(k)}}{\Delta f}\right) = S(r,\Delta f). \tag{28}$$

Then by (16), (27) and (28) we can have $T(r,\Delta f) = S(r,\Delta f)$ a contradiction.

Subcase 2.2. $P_2 = 0$. Then by (16), (27) and (28) and Lemma 1 we get

$$\begin{aligned} T(r,\Delta f) &\leqslant m\left(r, \frac{1}{\Delta f - P_1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,\Delta f) \\ &\leqslant m\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,\Delta f) \leqslant \left(T\left(r, f^{(k)}\right)\right) + S(r,\Delta f). \end{aligned} \tag{29}$$

From the fact that

$$T\left(r, f^{(k)}\right) \leqslant T(r,\Delta f) + S(r,\Delta f), \tag{30}$$

which follows from (29) that

$$T(r,\Delta f) = T\left(r, f^{(k)}\right) + S(r,\Delta f), \tag{31}$$

By second Nevanlinna Fundamental theorem, Lemma 1, (8) and (31) we have

$$\begin{aligned} 2T(r,\Delta f) &\leqslant 2T\left(r, f^{(k)}\right) + S(r,\Delta f) \\ &\leqslant \overline{N}\left(r, \frac{1}{f^{(k)}-P_1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-d}\right) + S(r,\Delta f) \\ &\leqslant \overline{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \overline{N}\left(r, \frac{1}{\Delta f}\right) + T\left(r, \frac{1}{f^{(k)}-d}\right) - m\left(r, \frac{1}{f^{(k)}-d}\right) + S(r,\Delta f) \\ &\leqslant 2T(r,\Delta f) - m\left(r, \frac{1}{f^{(k)}-d}\right) + S(r,\Delta f). \end{aligned}$$

Thus

$$m\left(r, \frac{1}{f^{(k)}-d}\right) = S(r, \Delta f). \tag{32}$$

From the First Fundamental Theorem, Lemma 1, (20) to (21), (31), (32) and Δf is a transcendental entire function of finite order, we obtain

$$\begin{aligned} m\left(r, \frac{\Delta f-d}{f^{(k)}-d}\right) &\leqslant m\left(r, \frac{\Delta f}{f^{(k)}-d}\right) + m\left(r, \frac{d}{f^{(k)}-d}\right) + S(r, \Delta f) \\ &\leqslant T\left(r, \frac{\Delta f-d}{f^{(k)}-d}\right) - N\left(r, \frac{\Delta f-d}{f^{(k)}-d}\right) + S(r, \Delta f) \\ &= m\left(r, \frac{f^{(k)}-d}{f}\right) + N\left(r, \frac{f^{(k)}-d}{f}\right) - N\left(r, \frac{\Delta f}{f^{(k)}-d}\right) + S(r, \Delta f) \\ \leqslant N\left(r, \frac{1}{\Delta f}\right) - N\left(r, \frac{1}{f^{(k)}-1}\right) &+ S(r, \Delta f) = T\left(r, \frac{1}{\Delta f}\right) - T\left(r, \frac{1}{f^{(k)}-d}\right) + S(r, \Delta f) \\ &= T(r, \Delta) - T\big(r, f^{(k)}\big) + S(r, \Delta f) = S(r, \Delta f). \end{aligned}$$

Thus, we get

$$m\left(r, \frac{\Delta f-d}{f^{(k)}-d}\right) = S(r, \Delta f). \tag{33}$$

It's easy to see that $N(r, \psi) = S(r, \Delta f)$ and ((12)) can be rewritten as

$$\psi = \left[\frac{P_1-d}{P_1}\frac{(f)^{(k+1)}}{f^{(k)}-P_1} + \frac{df^{(k+1)}}{f^{(k)}}\right]\left[\frac{f-d}{f^{(k)}-d} - 1\right]. \tag{34}$$

Then by (33) and (34) we can get

$$T(r, \psi) = m(r, \psi) + N(r, \psi) = S(r, \Delta f). \tag{35}$$

By (7), (25) and (35) we get

$$\overline{N}\left(r, \frac{1}{\Delta f - P_1}\right) = S(r, \Delta f). \tag{36}$$

Moreover, by (7), (31) and (36), we have

$$m\left(r, \frac{1}{f^{(k)}}\right) = S(r, \Delta f), \tag{37}$$

which implies

$$\overline{N}\left(r, \frac{1}{\Delta f}\right) = m\left(r, \frac{1}{\Delta f - P_1}\right) \leqslant m\left(r, \frac{1}{f^{(k)}}\right) = S(r, \Delta f). \tag{38}$$

Then by (7) we obtain $T(r, \Delta f) = S(r, \Delta f)$, a contradiction. So, by (12), (16), (27) and the Second Fundamental Theorem of Nevanlinna, we can get

$$\begin{aligned} T(r, \Delta f) &\leqslant 2m\left(r, \frac{1}{\Delta f - P_1}\right) + S(r, \Delta f) \leqslant 2m\left(r, \frac{1}{f^{(k)}}\right) + S(r, \Delta f) \\ \leqslant 2T\big(r, f^{(k)}\big) &- 2N\left(r, \frac{1}{f^{(k)}}\right) + S(r, \Delta f) \leqslant \overline{N}\left(r, \frac{1}{\Delta f - P_1}\right) + \overline{N}\left(r, \frac{1}{\Delta f - P_1}\right) \\ +\overline{N}\left(r, \frac{1}{f^{(k)}}\right) &- 2N\left(r, \frac{1}{f^{(k)}}\right) + S(r, \Delta f)T(r, \Delta f) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f), \end{aligned}$$

which deduces that

$$N\left(r, \frac{1}{f^{(k)}}\right) = S(r, \Delta f). \tag{39}$$

It follows from the second theorem of Nevanlinna that

$$\begin{aligned} T\big(r, f^{(k)}\big) &\leqslant \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - P_1}\right) + S(r, \Delta f) \\ &\leqslant \overline{N}\left(r, \frac{1}{f^{(k)} - P_1}\right) + S(r, \Delta f) \leqslant T\big(r, f^{(k)}\big) + S(r, \Delta f), \end{aligned}$$

which implies that

$$T\big(r, f^{(k)}\big) = \overline{N}\left(r, \frac{1}{f^{(k)} - P_1}\right) + S(r, \Delta f). \tag{40}$$

Similarly

$$T\big(r, f^{(k)}\big) = \overline{N}\left(r, \frac{1}{f^{(k)} - P_2}\right) + S(r, \Delta f). \tag{41}$$

Then, by (27), we get

$$T(r, \Delta f) = 2T\big(r, f^{(k)}\big) + S(r, \Delta f). \tag{42}$$

By (25) and (26) we have

$$T(r, \phi) = T\big(r, f^{(k)}\big) + S(r, \Delta f). \tag{43}$$

When case 1 occurs, we apply Lemma 4 and obtain

$$f^{(k)} = He^t. \tag{44}$$

Here $H \not\equiv 0$ is a small function of e^t. Rewrite (16) as

$$\phi = \frac{f^{(k+1)}(\Delta f - P_1)(\Delta f - P_2) - \Delta f\big(f^{(k)} - P_1\big)\big(f^{(k)} - P_2\big)}{(\Delta f - P_1)(\Delta f - P_2)\left(f^{(k)} - P_1\right)\left(f^{(k)} - P_2\right)}. \tag{45}$$

Combining (27) with (44) we get

$$X = f^{(k+1)}(\Delta f - P_1)(\Delta f - P_2) - \Delta f'\big(f^{(k)} - P_1\big)\big(f^{(k)} - P_2\big) = \sum_{i=0}^{5} \delta_i e^{it}, \tag{46}$$

and

$$Y = (\Delta f - P_1)(\Delta f - P_2) - \Delta f'\big(f^{(k)} - P_1\big)\big(f^{(k)} - P_2\big) = \sum_{j=0}^{5} \gamma_j e^{jt}, \tag{47}$$

where δ_i and γ_j are small functions of e^t, $\delta_5 \not\equiv 0$ and $\gamma_6 \not\equiv 0$.

If X and Y are two mutually prime polynomials in e^t, then we can get $T(r, \phi) = 6T(r, e^t) + S(r, \Delta f)$. It follows from (16), (41)–(43) that $T(r, \Delta f)$, a contradiction.

If X and Y are not two mutually prime polynomials in e^t, it's easy to see that the degree of Y is large than X. Then submitting (38) into (12) implies

$$H = P_2 \tag{48}$$

and

$$t = P_1 z + P_2, \tag{49}$$

where $P_1 \neq 0$ and P_2 are polynomials.

According to (45), (48), (49) and by simple calculation, we must have

$$\phi = \frac{C}{f^{(k)} - P_2}, \tag{50}$$

where C is a non-zero constant. Put (44) into (16) we have

$$\frac{c\left((f)^{(k)} - f^{(k+1)} - P_1\right)}{\left(f^{(k)} - P_1\right)\left(f^{(k)} - P_2\right)} = \frac{-\Delta f'}{(\Delta f - P_1)(\Delta f - P_2)}. \tag{51}$$

We claim that $f^{(k)} \equiv f^{(k+1)}$.

Otherwise, combining (22), (44) and (51) we can get $T(r, e^t) = S(r, \Delta f)$. It follows from (16) and (27) that $T(r, \Delta f) = S(r, \Delta f)$, a contradiction. Hence, it is a easy work to verify that

$$P_1 = 1 \tag{52}$$

and

$$f^{(k)} = P_2 e^{z - P_2} = Ae^z, \tag{53}$$

where A is a nonzero constant and furthermore

$$\Delta f = Ae^{2z} - P_1 Ae^z + P_1. \tag{54}$$

Then rewrite (27) as

$$\frac{\Delta f - f^{(k)}}{f^{(k)} - P_1} = e^t - 1. \tag{55}$$

Put (49), (52)–(54) into (55) and a direct calculation deduces

$$A = P_2 = e^{P_1} = 1. \tag{56}$$

It follows from (1), (28), (52) and (56) that

$$H = -P_1(e^{\varpi} - 1)^n = 1. \tag{57}$$

Since Δf and $f^{(k)}$ share P_2 IM and (41), (42) and (56) we get

$$e^{2z} - P_1 e^z + (P_1 - 1) = (e^z - 1)^2, \tag{58}$$

i. e.,

$$P_1 = 2. \tag{59}$$

It follows from (57) that

$$e^{\varpi} = (-2)^{-1/n} + 1. \tag{60}$$

But we cannot get (2) from (60), a contradiction. When case 2 occurs we know that $m(r, e^t) = m(r, e^Q) + O(1) = S(r, \Delta f)$. Then by (16) and (27) we deduce $T(r, \Delta f) = S(r, \Delta f)$ a contradiction. $\triangleright$

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ЕДИНСТВЕННОСТЬ ЦЕЛЫХ ФУНКЦИЯХ ОТНОСИТЕЛЬНО ИХ РАЗНОСТНЫХ ОПЕРАТОРОВ И ПРОИЗВОДНЫХ

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Аннотация. В этой статье мы изучаем единственность целых функций относительно их разностного оператора и производных. Представление о целых и мероморфных функциях сильно зависит от этого направления. Рубель и Янг рассмотрели единственность целой функции и ее производных; они доказали, что если $f(z)$ и $f'(z)$ разделяют два значения a, b с учетом кратностей, то $f(z) \equiv f'(z)$. Позже Ли Пинг и Янг улучшили результат Рубеля и Янга: если $f(z)$ — непостоянная целая функция, а a и b — два конечных различных комплексных значения, и если $f(z)$ и $f^{(k)}(z)$ разделяют a с учетом кратностей и b — без учета кратностей, то $f(z) \equiv f^{(k)}(z)$. В последние годы проявляется значительный интерес к распределению значений мероморфных функций конечного порядка относительно разностного аналога. Заменив различные конечные комплексные значения многочленами, устанавливается следующий результат: пусть $\Delta f(z)$ — трансцендентная целая функция конечного порядка, $k \geqslant 0$ — целое число, а P_1 и P_2 — два многочлена; если $\Delta f(z)$ и $f^{(k)}$ разделяют P_1 с учетом кратностей и P_2 игнорируя кратности, то $\Delta f \equiv f^{(k)}$. Нетривиальное доказательства этого результата использует теорию распределения значений Неванлинны.

Ключевые слова: разностный оператор, разделяемые значения, конечный порядок, единственность, целая функция, многочлены.

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