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ON THE WEAK CONVERGENCE OF OPERATORS ITERATIONS IN VON NEUMANN ALGEBRAS

A. A. Katz

Equivalent conditions are obtained for the weak convergence of iterations of the positive contractions in the pre-conjugate spaces of von Neumann algebras.

1. Introduction

This paper is devoted to ergodic type properties of the weak convergence of operators iterations in von Neumann algebras.

The first results in the field of non-commutative ergodic theory were obtained by Sinai and Anshelevich [18] and Lance [14]. Developments of the subject are reflected in the monographs of Jajte [9] and Krengel [13] (see also [4]–[7], [10], [11], [16]).

We will use facts and the terminology from the general theory of von Neumann algebras ([1], [2], [15], [17], [19]).

Let M be a von Neumann algebra, acting on a separable Hilbert space H, M_* is a pre-conjugate space of M, which always exists according to the Sakai theorem [17].

Recall some standard terminology ([4], [5], [6], [10], [11], [13]).

Definition 1. A linear mapping T from M_* in itself is called a *contraction* if its norm is not greater then one.

Definition 2. A contraction T is said to be *positive* if $TM_{*+} \subset M_{*+}$.

We will consider two topologies on the space M_*: the *weak topology*, or the $\sigma(M_*, M)$ topology, and the *strong topology* of the M_*-space norm convergence.

Definition 3. A matrix $(a_{n,i})$, $i, n = 1, 2, \dots$ of real numbers is called *uniformly regular*, if:

$$\sup_n \sum_{i=1}^{\infty} |a_{n,i}| \leqslant C < \infty, \quad \lim_{n\to\infty} \sup_i |a_{n,i}| = 0, \quad \lim_{n\to\infty} \sum_i a_{n,i} = 1.$$

2. Main Result

The following theorem is valid:

Theorem 1. *The following conditions for a positive contraction T in the pre-conjugate space of a von Neumann algebras M are equivalent:*

(i) *The sequence $\{T^i\}_{i=1,2,\dots}$ converges weakly;*

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(ii) *For each strictly increasing sequence of natural numbers* $\{k_i\}_{i=1,2,\dots}$,

$$n^{-1}\sum_{i<n} T^{k_i},$$

converges strongly;

(iii) *For any uniformly regular matrix* $(a_{n,i})$, *the sequence* $\{A_n(T)\}_{n=1,2,\dots}$,

$$A_n(T)=\sum_i a_{n,i}(T^i),$$

converges strongly.

◁ We first prove the following lemma:

Lemma 1. *Let there exists a uniformly regular matrix* $(a_{n,i})$ *such that for each strictly increasing sequence* $\{k_i\}_{i=1,2,\dots}$ *of natural numbers,*

$$B_n=\sum_i a_{n,i}T^{k_i}, \tag{1}$$

converges strongly. Then the sequence $\{T^i\}_{i=1,2,\dots}$ *converges weakly.*

◁ Let $(a_{n,i})$ be a matrix with the aforementioned properties. Then the limit B_n is not dependent upon the choice of the sequence $\{k_i\}_{i=1,2,\dots}$. In fact, let $\{k_i\}_{i=1,2,\dots}$ and $\{l_i\}_{i=1,2,\dots}$ be the sequences for which the limits B_n are different. This means that for $x\in M_*$,

$$\sum_i a_{n,i}T^{k_i}x\to x_1,$$

and

$$\sum_i a_{n,i}T^{l_i}x\to x_2,$$

as $n\to\infty$. For a matrix $(a_{n,i})$ let us build increasing sequences $\{i_j\}_{j=1,2,\dots}$ and $\{n_j\}_{j=1,2,\dots}$, such that

$$\lim_{j\to\infty}\left[\sum_{i<i_{j-1}}\left|a_{n_j,i}\right|+\sum_{i>i_j}\left|a_{n_j,i}\right|\right]=0.$$

Let

$$m_i=k_i \text{ for } i\in[i_{2j-1},i_{2j}) \text{ and } m_i=l_i \text{ for } i\in[i_{2j},i_{2j+1}),\quad j=1,2,\dots$$

Then

$$\lim_j\left\|\sum_i a_{n_{2j+1},i}T^{m_i}x-x_1\right\|=0,\quad \lim_j\left\|\sum_i a_{n_{2j},i}T^{m_i}x-x_2\right\|=0,$$

which contradicts (1), and therefore $x_1=x_2$. Let now $y\in M$ is such that

$$(T^n x-x_1,y)\to 0,$$

when $n\to\infty$. Let us choose a subsequence $\{k_i\}$ such that

$$(T^{k_i}x-x_1,y)\to\gamma\neq 0,$$

where γ is a real number. Then, from the uniform regularity of the matrix $(a_{n,i})$ it follows that

$$\lim_n\left(\sum_i a_{n,i}T^{k_i}x-x_1,y\right)=\gamma,$$

which contradicts the choice of the matrix $(a_{n,i})$. $\triangleright$

[Proof of the Theorem 1 (cont.).] Because the implication (ii) $\Longrightarrow$ (iii) is obvious, the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) immediately follow from the lemma 1.

The implication (iii) $\Longrightarrow$ (ii) is trivial, because the matrix $(a_{n,i})$, $a_{n,i} = \frac{1}{n} \sum\limits_{i<n} \delta_{j,k_i}$ is uniformly regular.

Applying the above with lemma 1, $a_{n,i} = \frac{1}{n}$, $i \leqslant n$ and $a_{n,i} = 0$ for $i > n$, we get the implication (ii) $\Longrightarrow$ (i).

To prove the implication (i) $\Longrightarrow$ (ii), we would need the following lemma:

Lemma 2. *Let Q be a contraction in the Hilbert space H. Then the weak convergence of $Q^n x$ in H, where $x \in H$, implies the strong convergence of*

$$\sum_i a_{n,i} Q^n x$$

for any uniformly regular matrix $(a_{n,i})$.

$\triangleleft$ If the weak limit $Q^n x$ exists and is equal to x_1, then

$$Qx_1 = Q(\lim_{n\to\infty} Q^n x) = x_1,$$

where the limit is considered in the weak topology, i. e. x_1 is Q-invariant. Therefore

$$\left\|\sum_i a_{N,i} Q^i x\right\|^2 \leqslant \sum_i \sum_j a_{N,i} a_{N,j} (Q^i x, Q^j x) \leqslant \sum_i \sum_j \left|a_{N,i} a_{N,j} (Q^i x, Q^j x)\right|.$$

Let us fix $\varepsilon > 0$. Because Q is a contraction, the limit $\|Q^n x\|$ does exist. Now, we can find $K > 0$, such that for $k > K$ and $j \geqslant 0$,

$$\left\|Q^k x\right\| - \left\|Q^{k+j} x\right\| \leqslant \varepsilon^2$$

and

$$\left|(Q^k x, x)\right| \leqslant \varepsilon.$$

Then,

$$\begin{aligned}
\left|(Q^k x, x) - (Q^{k+j} x, Q^j x)\right| &= \left|(Q^k x, x) - (Q^{*j} Q^{k+j} x, x)\right| \\
&\leqslant \left\|Q^k x - Q^{*j} Q^{k+j} x\right\| \cdot \|x\| = \left(\left\|Q^k x - Q^{*j} Q^{k+j}\right\|^2\right)^{\frac{1}{2}} \cdot \|x\| \\
&= \left(\left\|Q^k x\right\|^2 - 2\left\|Q^{k+j} x\right\|^2 + \left\|Q^{*j} Q^{k+j} x\right\|^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\left\|Q^k x\right\|^2 - \left\|Q^{k+j} x\right\|^2\right) \cdot \|x\| \leqslant \varepsilon \cdot \|x\|,
\end{aligned}$$

and therefore

$$\left|(Q^{k+j} x, Q^j x)\right| \leqslant \varepsilon \cdot (1 + \|x\|)$$

for all $k > K$ and $j \geqslant 0$, or for $|i - j| \geqslant k$ the inequality

$$\left|(Q^i x, Q^j x)\right| \leqslant \varepsilon \cdot (1 + \|x\|),$$

is valid. We will fix $\eta > 0$, and let N be such a natural number with $\max_i |a_{n,i}| < \eta$, for $n \geqslant N$. Then the expression (1) for $n \geqslant N$ could be estimated the following way:

$$\begin{aligned}\sum_i \sum_j \left|a_{N,i}a_{N,j}(Q^i x, Q^j x)\right| &= \sum_{|i-j|\leqslant k} \left|a_{n,i}a_{n,j}(Q^i x, Q^j x)\right| + \sum_{|i-j|>k} \left|a_{n,i}a_{n,j}(Q^i x, Q^j x)\right| \\ &\leqslant \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + \sum_i \sum_j |a_{n,i}a_{n,j}| \cdot \varepsilon \cdot (1 + \|x\|) \\ &\leqslant C \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + C^2 \cdot \varepsilon \cdot (1 + \|x\|).\end{aligned}$$

From the arbitrarity of the values of ε and η it follows that the strong convergence is present and the lemma is proven. $\rhd$

[Proof of the Theorem 1 (cont.).] Let us prove the implication (i) $\Longrightarrow$ (iii). Let $x \in M_{*+}$ and the sequence $\{T^i x\}_{i=1,2,\ldots}$ converges weakly. Without loss of generality we can consider $\|x\| \leqslant 1$, and let

$$\overline{x} = \lim_{n\to\infty} T^n x,$$

where the limit is understood in the weak sense. Let us consider

$$y = \sum_{n=0}^{\infty} 2^{-n} T^n x.$$

The series that defines y is convergent in the norm of the space M_*. From the positivity of x and the properties of the operator T it follows that $Ty \leqslant 2y$, and, therefore, for all $k = 1, 2, \ldots$, $s(T^k y) \leqslant s(y)$, where by $s(z)$ we denote the support of the normal functional z.

Lemma 3. *Let $u \in M_{*+}$ and $s(u) \leqslant s(y)$. Then $s(\overline{u}) \leqslant s(\overline{x})$, where*

$$\overline{u} = \lim_{n\to\infty} T^n u.$$

$\lhd$ Indeed, let us fix $\varepsilon > 0$. From the density of the set

$$\mathfrak{L} = \{w \in M_{*+},\ w \leqslant \lambda y,\ \text{for some} \lambda > 0\},$$

in the set

$$\mathfrak{S} = \{w \in M_{*+},\ s(w) \leqslant s(y)\},$$

in the norm of the space M_* it follows that there are $\lambda > 0$ and $w \in \mathfrak{L}$ such that

$$\|w - u\| \leqslant \varepsilon \text{ and } w \leqslant \lambda y.$$

Let $\overline{w} = \lim_{n\to\infty} T^n w$. Then

$$\begin{aligned}\overline{w}(\mathbf{1}-s(\overline{x})) &= \lim_{n\to\infty} (T^n(w))(\mathbf{1}-s(\overline{x})) \leqslant \lambda \cdot \lim_{n\to\infty} (T^n y)(\mathbf{1}-s(\overline{x})) \\ &\leqslant \lambda \cdot \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} 2^{-k} \cdot (T^{n+k} x)(\mathbf{1}-s(\overline{x})) \right) \\ &= \lambda \cdot \sum_{k=0}^{\infty} 2^{-k} \lim_{n\to\infty} (T^{n+k} x)(\mathbf{1}-s(\overline{x})) = 0.\end{aligned}$$

Because the operator T does not increase the norm of the functionals from M_*, we get that

$$\overline{u}(\mathbf{1}-s(\overline{x})) = \lim_{n\to\infty}(T^n u)(\mathbf{1}-s(\overline{x})) \leqslant \lim_{n\to\infty}(T^n w)(\mathbf{1}-s(\overline{x})) + \lim_{n\to\infty}\|T^n(w-u)\| \leqslant \varepsilon.$$

The needed inequality follows, since ε is arbitrary. ▷

[Proof of the Theorem 1 (cont.).] Let now $\mu \in M_*$. We will denote by $\mu.E$, where E is a projection from the algebra M, the functional

$$(\mu.E)(A) = \mu(EAE),$$

where $A \in M$. Let us fix $\varepsilon > 0$. We will find a number N, such that

$$(T^n x)(\mathbf{1}-s(\overline{x})) < \varepsilon^2$$

for $n > N$. Thus,

$$\left\|T^N x.s(\overline{x}) - T^N x\right\| = \sup_{\substack{A\in M\\ \|A\|_\infty\leqslant 1}} \big|(T^N x)((\mathbf{1}-s(\overline{x}))A(\mathbf{1}-s(\overline{x})))$$

$$+(T^N x)((s(\overline{x}))A(\mathbf{1}-s(\overline{x}))) + (T^N x)((\mathbf{1}-s(\overline{x}))A(s(\overline{x})))\big| \leqslant \varepsilon\cdot(\varepsilon + 2\,\|x\|^{\frac{1}{2}}),$$

because

$$|\mu(AB)|^2 \leqslant \mu(A^*A)\cdot\mu(B^*B),$$

where $\mu \in M_{*+}$ and $A, B \in M$.

Let $w \in \mathfrak{L}$ is such that $w \leqslant \lambda\overline{x}$ for some $\lambda > 0$ and $\left\|T^N x.s(\overline{x}) - w\right\| \leqslant \varepsilon$. Then, for $n > N$, the following is valid:

$$\begin{aligned}\left\|T^n x - T^{n-N} w\right\| &\leqslant \left\|T^{n-N}(T^N x - T^N x.s(\overline{x}))\right\| \\ &+ \left\|T^{n-N}(T^N x.s(\overline{x}) - w)\right\| \leqslant 4\cdot\varepsilon.\end{aligned} \tag{2}$$

By taking the weak limit in the inequality (2) and because the unit ball of M_* is weakly closed, we will get $\|\overline{x} - \overline{w}\| \leqslant 4\cdot\varepsilon$, where $\overline{w} = \lim_{n\to\infty} T^n w$.

Let us now consider the algebra $M_{s(x)}$. The functional $\overline{x}$ is faithful on the algebra $M_{s(x)}$. We will consider the representation $\pi_{\overline{x}}$ of the algebra $M_{s(x)}$ constructed using the functional x [2]. Because the functional $\overline{x}$ is faithful, we can conclude that the representation $\pi_{\overline{x}}$ is faithful on the algebra $M_{s(\overline{x})}$, and therefore $\pi_{\overline{x}}$ is an isomorphism of the algebra $M_{s(\overline{x})}$ and some algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is a von Neumann algebra, and its pre-conjugate space $\mathfrak{A}_*$ is isomorphic to the space $M_*.s(\overline{x})$ ([17]). Let us note now that

$$TM_*.s(\overline{x}) \subset M_*.s(\overline{x}).$$

In fact, $T\mathfrak{L} \subset \mathfrak{L}$, and therefore, by taking the norm closure we get $TS \subset S$; by taking now the linear span we will get

$$TM_*.s(\overline{x}) \subset M_*.s(\overline{x}).$$

Denote by $\overline{T}$ the isomorphic image of the operator T, acting on the space $\mathfrak{A}_*$. Let $u \in \mathfrak{A}_{*+}$ and $u \leqslant \lambda\overline{x}$ for some $\lambda > 0$. Then there exists the operator $B \in \mathfrak{A}'$, where $\mathfrak{A}'$ is a commutant of $\mathfrak{A}$, such that $(AB\Omega, \Omega) = u(A)$ for all $A \in \mathfrak{A}$. Note, that from the lemma 2

$$(\overline{T}u)(A) = u((\overline{T})^*A) = (((\overline{T})^*A)B\Omega, \Omega) = (A((\overline{T}^*)'B)\Omega, \Omega).$$

Also, from

$$\overline{T}\mathfrak{A}_{*+} \subset \mathfrak{A}_{*+}, \left\|\overline{T}u\right\| \leqslant \|u\| \quad \text{and} \quad \overline{T}\overline{x} = \overline{x}$$

it follows that

$$(\overline{T})^{*}\mathfrak{A}_{+}; (\overline{T}^{*})\mathbf{1} \leqslant \mathbf{1} \quad \text{and} \quad \left\|(\overline{T})^{*}A\right\|_{\infty} \leqslant \|A\|_{\infty}$$

for all $A \in \mathfrak{A}$. Based on the lemma we now conclude that

$$\left\|(\overline{T}^{*}B)\right\|_{\infty} \leqslant \|B\|_{\infty}; \ \overline{T}^{*\prime}\mathfrak{A}'_{+} \subset \mathfrak{A}'_{+}; \ \overline{T}^{*\prime}\mathbf{1} \leqslant \mathbf{1}$$

for all $B \in \mathfrak{A}'$.

The space $\mathfrak{A}'_{sa}$ is a pre-Hilbert space of the self adjoint operators from $\mathfrak{A}'$ with the scalar product $(B, C)_{\overline{x}} = (CB\Omega, \Omega)$, and, using the Kadison inequality [1] we have

$$((\overline{T}^{*\prime}B)(\overline{T}^{*\prime}B)\Omega, \Omega) \leqslant (\overline{T}^{*\prime}(B^{2})\Omega, \Omega) \leqslant (B\Omega, B\Omega),$$

i. e. the operator $\overline{T}^{*\prime}$ is a contraction in the pre-Hilbert space $(\mathfrak{A}'_{sa}, (\cdot,\cdot)_{\overline{x}})$.

We will identify $M_{*}.s(\overline{x})$ and $\mathfrak{A}_{*}$. Because $w \in \mathfrak{L}$, i. e. $w \leqslant \lambda\overline{x}$ for some $\lambda > 0$, then $\overline{w} \leqslant \lambda\overline{x}$ as well. Let

$$w(A) = (BA\Omega, \Omega) \text{ and } \overline{w}(A) = (\overline{B}A\Omega, \Omega)$$

for all $A \in \mathfrak{A}$, where B, $\overline{B} \in \mathfrak{A}'$.

Let now $(a_{n,i})$ be a uniformly regular matrix. Using lemma 2 we will find $k \in \mathbb{N}$ so that

$$\left\|\sum_{i} a'_{k,i}T^{i}w - \overline{w}\right\| = \sup_{\substack{A\in\mathfrak{A} \\ \|A\|_{\infty}=1}} \left|\left(\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^{i}(A - \overline{B})A\Omega, \Omega\right)\right|$$

$$\leqslant \left(\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^{i}(B - \overline{B})\Omega \sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^{i}(B - \overline{B})\right)^{\frac{1}{2}} \cdot \sup_{\substack{A\in\mathfrak{A} \\ \|A\|_{\infty}\leqslant 1}} (A\Omega, A\Omega)^{\frac{1}{2}}$$

$$\leqslant (\overline{x}(\mathbf{1}))^{\frac{1}{2}} \cdot \left\|\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^{i}(B - \overline{B})\right\|_{(.,.)_{\overline{x}}} < \varepsilon$$

for $k > K$, where by $(a'_{n,i})$ we will denote a matrix with the elements

$$a'_{n,i} = \left(\sum_{i>N} a_{n,j}\right)^{-1} a_{n,j+N}.$$

It is easy to see that the matrix $(a'_{n,i})$ will be uniformly regular as well.

Then, for a big enough $k > K$ we will have

$$\left\|\sum_{i} a_{k,i}T^{i}x - \overline{x}\right\| \leqslant \sum_{i\leqslant N} |a_{k,i}| \left\|T^{i}x - \overline{x}\right\| + \sum_{i>N} |a_{k,i}| \left\|T^{i}x - T^{i-N}w\right\|$$

$$+ \sum_{i>N} |a_{k,i}| \cdot \left|1 - \left(\sum_{i>N} a_{k,i}\right)^{-1}\right| \left\|T^{i-N}w\right\| + \left\|\sum_{j=1}^{\infty} a_{k,j+N} \cdot \left(\sum_{i>N} a_{k,i}\right)^{-1} T^{j}w - \overline{w}\right\|$$

$$+ \left\|\left(\sum_{i\leqslant N} a_{k,i}\right) \cdot \overline{w}\right\| + \left|\sum_{i>N} a_{k,i}\right| \|\overline{w} - \overline{x}\|$$

$$\leqslant \sum_{i\leqslant N} 2\cdot\frac{\varepsilon}{N} + \sum_{i>N} |a_{k,i}|\cdot 4\varepsilon + \sum_{i>N} |a_{k,i}|\left(1-(1+\varepsilon)^{-1}\right)\cdot 2 + \sum_{i\leqslant N} 2\cdot\frac{\varepsilon}{N} + (1+\varepsilon)\cdot 4\varepsilon$$

$$\leqslant 2\varepsilon + (1+\varepsilon)\cdot 4\varepsilon + \varepsilon\cdot 2\cdot(1+\varepsilon) + \varepsilon + 2\varepsilon + (1+\varepsilon)\cdot 4\varepsilon \leqslant 25\varepsilon.$$

The arbitrarity of ε proves the needed statement. The proof of the theorem is now completed. $\rhd$

Remark. Extention of the discussed properties of iterations to non-commutative L_p-spaces will be separately presented in the forthcoming paper [12].

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Prof. Alexander A. Katz, Ph. D.
Department of Mathematics and Computer Science,
St. John's University, NY 10301, USA
E-mail: `katza@stjohns.edu`