Vladikavkaz Mathematical Journal
2018, Volume 20, Issue 2, P. 23–28

УДК 517.98
DOI 10.23671/VNC.2018.2.14715

DERIVATIONS ON BANACH $*$-IDEALS IN VON NEUMANN ALGEBRAS

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Abstract. It is known that any derivation $\delta : \mathscr{M} \to \mathscr{M}$ on the von Neumann algebra $\mathscr{M}$ is an inner, i. e. $\delta(x) := \delta_a(x) = [a, x] = ax - xa$, $x \in \mathscr{M}$, for some $a \in \mathscr{M}$. If H is a separable infinite-dimensional complex Hilbert space and $\mathscr{K}(H)$ is a C^*-subalgebra of compact operators in C^*-algebra $\mathscr{B}(H)$ of all bounded linear operators acting in H, then any derivation $\delta : \mathscr{K}(H) \to \mathscr{K}(H)$ is a spatial derivation, i.e. there exists an operator $a \in \mathscr{B}(H)$ such that $\delta(x) = [x, a]$ for all $x \in K(H)$. In addition, it has recently been established by Ber A. F., Chilin V. I., Levitina G. B. and Sukochev F. A. (JMAA, 2013) that any derivation $\delta : \mathscr{E} \to \mathscr{E}$ on Banach symmetric ideal of compact operators $\mathscr{E} \subseteq \mathscr{K}(H)$ is a spatial derivation. We show that the same result is also true for an arbitrary Banach $*$-ideal in every von Neumann algebra $\mathscr{M}$. More precisely: If $\mathscr{M}$ is an arbitrary von Neumann algebra, $\mathscr{E}$ be a Banach $*$-ideal in $\mathscr{M}$ and $\delta \colon \mathscr{E} \to \mathscr{E}$ is a derivation on $\mathscr{E}$, then there exists an element $a \in \mathscr{M}$ such that $\delta(x) = [x, a]$ for all $x \in \mathscr{E}$, i. e. δ is a spatial derivation.

Key words: von Neumann algebra, Banach $*$-ideal, derivation, spatial derivation.
Mathematical Subject Classification (2010): 46L57, 46L51, 46L52.

1. Introduction

It is well known [1, Lemma 4.1.3] that every derivation on a C^*-algebra A is norm continuous. In fact, this also easily follows from another well known fact [1, Corollary 4.1.7] that every derivation on A realized as a $*$-subalgebra in the algebra $\mathscr{B}(H)$ of all bounded linear operators on a Hilbert space H is given by a reduction of an inner derivation on a von Neumann algebra $\mathscr{M} = \overline{A}^{wo}$ (the closure of A in the weak operator topology on $\mathscr{B}(H)$). In the special setting when $A = \mathscr{K}(H)$ (the ideal of all compact operators on H) and $\mathscr{M} = \mathscr{B}(H)$, the latter result states that for every derivation δ on A there exists an operator $a \in \mathscr{B}(H)$ such that $\delta(x) = [a, x]$ for every $x \in \mathscr{K}(H)$. The ideal $\mathscr{K}(H)$ is a classical example of a Banach operator ideal in $\mathscr{B}(H)$ (see [2, 3, 4, 5]). Any such ideal $\mathscr{E} \neq \mathscr{K}(H)$ is a Banach $*$-algebra (albeit not a C^*-algebra) and a natural question immediately suggested by this discussion is as follows.

Question 1. *Let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}}) \subseteq \mathscr{K}(H)$ be a Banach ideal of compact operators on H and let $\delta \colon \mathscr{E} \to \mathscr{E}$ be a derivation on $\mathscr{E}$. Is δ continuous with respect to a norm $\|\cdot\|_{\mathscr{E}}$ on $\mathscr{E}$? If this fact is true, then does there exist an operator $a \in \mathscr{B}(H)$ such that $\delta(x) = [a, x]$ for every $x \in \mathscr{E}$?*

The positive answer to Question 1 was obtained in the paper [6] (see also [7]).

Let now $\mathscr{M}$ be an arbitrary von Neumann algebra. An $*$-ideal $\mathscr{E}$ of $\mathscr{M}$ is called *a Banach $*$-ideal*, if $\mathscr{E}$ is equipped with a Banach norm $\|\cdot\|_{\mathscr{E}}$, such that

$$\|axb\|_{\mathscr{E}} \leqslant \|a\|_{\mathscr{M}} \cdot \|x\|_{\mathscr{E}} \cdot \|b\|_{\mathscr{M}}$$

for all $x \in \mathscr{E}$ and $a, b \in \mathscr{M}$.

It is natural to pose the following variant of question 1.

Question 2. *Let $\mathscr{M}$ be an arbitrary von Neumann algebra and let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a Banach $*$-ideal of $\mathscr{M}$. Let $\delta\colon \mathscr{E} \to \mathscr{E}$ be a derivation on $\mathscr{E}$. Is δ continuous with respect to a norm $\|\cdot\|_{\mathscr{E}}$ on $\mathscr{E}$? If this fact is true, then does there exist an operator $a \in \mathscr{M}$ such that $\delta(x) = [a, x]$ for every $x \in \mathscr{E}$?*

The following theorem, the main result of this paper, gives a positive answer to Question 2.

Theorem 1. *Let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a Banach $*$-ideal of the von Neumann algebra $\mathscr{M}$ and let $\delta\colon \mathscr{E} \to \mathscr{E}$ be a derivation on $\mathscr{E}$. Then there exists an element $a \in \overline{\mathscr{E}}^{wo}$ such that $\delta(x) = [a, x]$ for all $x \in \mathscr{E}$. Moreover, we can choose such an element a as follows: $\|a\|_{\mathscr{M}} \leqslant \|\delta\|_{\mathscr{E}\to\mathscr{E}}$.*

2. Preliminaries

For details on the von Neumann algebra theory, the reader is referred to e. g. [1, 8, 9].

Let H be a Hilbert space over the field $\mathbb{C}$ of complex numbers, let $\mathscr{B}(H)$ be the $*$-algebra of all bounded linear operators on H, let $\mathscr{M}$ be a von Neumann subalgebra in $\mathscr{B}(H)$ and let $\mathscr{P}(\mathscr{M}) = \{p \in \mathscr{M} : p^2 = p = p^*\}$ be the lattice of all projections in $\mathscr{M}$. The center of a von Neumann algebra $\mathscr{M}$ will be denoted by $\mathscr{Z}(\mathscr{M})$.

Let A be an arbitrary subalgebra in $\mathscr{M}$. A linear mapping $\delta\colon A \to \mathscr{M}$ is called *derivation* on A with values in $\mathscr{M}$ if the equality $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in A$. It is not difficult to verify that for every $a \in A$ the mapping $\delta_a(x) = [a, x] = ax - xa$, $x \in A$, defines a derivation on A, in addition $\delta_a(A) \subseteq A$. Such derivations δ_a are called *inner derivations* on A.

If A is a $*$-subalgebra in $\mathscr{M}$ then a derivation $\delta\colon A \to \mathscr{M}$ is said to be a *$*$-derivation* if $\delta(x^*) = \delta(x)^*$ for all $x \in A$. For every derivation $\delta\colon A \to \mathscr{M}$ of a $*$-algebra A into $\mathscr{M}$ we define mappings

$$\delta_{\mathrm{Re}(x)} := \frac{\delta(x) + \delta(x^*)^*}{2}, \qquad \delta_{\mathrm{Im}}(x) := \frac{\delta(x) - \delta(x^*)^*}{2i}, \quad x \in A.$$

It is easy to see that δ_{Re} and δ_{Im} are $*$-derivations on A, moreover $\delta = \delta_{\mathrm{Re}} + i\delta_{\mathrm{Im}}$.

Let $\mathscr{E}$ be a two-sided ideal in $\mathscr{M}$. Then $\mathscr{E}$ is an $*$-ideal in $\mathscr{M}$ and the conditions $x \in \mathscr{M}$, $y \in \mathscr{E}$, $|x| \leqslant |y|$ imply that $x \in \mathscr{E}$.

We need the following property of two-sided ideals in von Neumann algebras.

Proposition 1 [10, Proposition 2.4.22]**.** *If $\mathscr{E}$ is wo-closed two-sided ideal in a von Neumann algebra $\mathscr{M}$ then there exists a central projection $z \in \mathscr{Z}(\mathscr{M}))$ such that $\mathscr{E} = z \cdot \mathscr{M}$.*

A non-zero two-sided ideal $\mathscr{E}$ of $\mathscr{M}$, equipped with a Banach norm $\|\cdot\|_{\mathscr{E}}$, is called *a Banach $*$-ideal*, if

$$\|axb\|_{\mathscr{E}} \leqslant \|a\|_{\mathscr{M}} \cdot \|b\|_{\mathscr{M}} \cdot \|x\|_{\mathscr{E}}$$

whenever $x \in \mathscr{E}$ and $a, b \in \mathscr{M}$.

It should be observed that any a Banach $*$-ideal $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ is $*$-closed and that $x \in \mathscr{M}$, $y \in \mathscr{E}$ and $|x| \leqslant |y|$ imply that $x \in \mathscr{E}$ and $\|x\|_{\mathscr{E}} \leqslant \|y\|_{\mathscr{E}}$.

Let A be a C^*-subalgebra in the C^*-algebra $\mathscr{B}(H)$. By [1, Lemma 4.1.3] every derivation $\delta\colon A \to A$ is a $\|\cdot\|_{\mathscr{B}(H)}$-continuous. The following Theorem gives an extension of the derivation δ to the von Neumann algebra $\overline{A}^{wo}$, where $\overline{A}^{wo}$ is a wo-closure of C^*-subalgebra A in $\mathscr{B}(H)$.

Theorem 2 [10, Proposition 3.2.24], [1, Theorem 4.1.6, Corollary 4.1.7], [11, Theorem 2]**.** *Let A be a C^*-subalgebra in the C^*-algebra $\mathscr{B}(H)$ and let $\delta\colon A \to A$ be a derivation on A. Then there exists an element a in $\overline{A}^{wo} = \mathscr{N}$ such that $\delta(x) = \delta_a(x) = [a, x]$ for all $x \in A$ and $\|\delta\|_{A\to A} = \|\delta_a\|_{\mathscr{N}\to\mathscr{N}}$. Moreover we can choose such an element $a \in \mathscr{N}$ as follows: $\|a\|_{\mathscr{N}} \leqslant \frac{1}{2} \cdot \|\delta_a\|_{\mathscr{N}\to\mathscr{N}}$.*

3. Main Results

Throughout this section $\mathscr{M}$ is an arbitrary von Neumann algebra. We recall that a projection $p \in \mathscr{P}(\mathscr{M})$ is called *an atom* if $0 \neq q \in \mathscr{P}(\mathscr{M})$, $q \leqslant p$ imply that $q = p$. If q is an atom then $q \cdot \mathscr{M} \cdot q = q \cdot \mathbb{C}$.

Proposition 2. *Let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a Banach ∗-ideal in the von Neumann algebra $\mathscr{M}$ and let $\delta\colon \mathscr{E} \to \mathscr{E}$ be a derivation on $\mathscr{E}$. Then δ is a continuous mapping on $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$.*

◁ Without loss of generality, we may assume that δ is a ∗-derivation. Since $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ is a Banach space, it is sufficient to prove that the graph of δ is closed. Suppose a contrary. Then there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathscr{E}$ and an element $0 \neq a \in \mathscr{E}$ such that $a = a^*$, $\|a_n\|_{\mathscr{E}} \to 0$ and $\|\delta(a_n) - a\|_{\mathscr{E}} \to 0$ as $n \to \infty$.

Let $a = a_+ - a_-$ be an orthogonal decomposition of a, that is $a_+, a_- \in \mathscr{E}$, $a_+, a_- \geqslant 0$, and $a_+ a_- = 0$. Without loss of generality, we may assume that $a_+ \neq 0$, otherwise we consider the sequence $\{-a_n\}_{n=1}^{\infty}$. Since $a \in \mathscr{E}$, there exists a projection $p \in \mathscr{M}$ such that $pap \geqslant \lambda p$ for some $\lambda > 0$. Replacing a_n with $\frac{a_n}{\lambda}$ we may assume $pap \geqslant p$. Hence, for some operator $c \in \mathscr{M}$, we have $p = c^* papc \in \mathscr{E}$.

There are two possible cases:

(i) There exists an atom $0 \neq q \in \mathscr{P}(\mathscr{M})$ such that $q \leqslant p$;

(ii) The lattice $\mathscr{P}(\mathscr{M})$ does not contain atoms $q \neq 0$ such that $q \leqslant p$.

In the case (i), we have $q \in \mathscr{E}$ and $q \leqslant qaq$. Since q is an atom, it follows $qa_nq = \lambda_n q$, $\lambda_n \in \mathbb{C}$, and we immediately deduce that $\lim_{n\to\infty} \lambda_n = 0$ from the assumption $\|a_n\|_{\mathscr{E}} \to 0$. Since

$$\delta(qa_nq) = \delta(q)a_nq + q\delta(a_nq)) = \delta(q)a_nq + q\delta(a_n)q + qa_n\delta(q)$$

it follows that

$$\|\delta(qa_nq) - q\delta(a_n)q\|_{\mathscr{E}} \leqslant 2\|\delta(q)\|_{\mathscr{M}}\|a_n\|_{\mathscr{E}} \to 0, \quad \text{as } \ n \to \infty,$$

and

$$q \leqslant qaq = \|\cdot\|_{\mathscr{E}} - \lim_{n\to\infty} \delta(qa_nq) = \|\cdot\|_{\mathscr{E}} - \lim_{n\to\infty} \delta(\lambda_n q) = \delta(q) \lim_{n\to\infty} \lambda_n = 0.$$

This contradicts with the assumption that $q \neq 0$.

In the case (ii), there exists a pairwise orthogonal sequence $\{e_n\}_{n=1}^{\infty} \subset \mathscr{P}(\mathscr{M})$ such that $0 \neq e_n \leqslant p$ for all $n \geqslant 1$. Clearly, we have $\{e_n\}_{n=1}^{\infty} \subset \mathscr{E}$ and $e_nae_n \geqslant e_n$ for any natural number $n \in \mathbb{N}$. Let $\{m_n\}_{n=1}^{\infty}$ be any sequence of positive integers such that

$$m_n > (2n+1)/\|e_n\|_{\mathscr{E}}, \quad n \geqslant 1.$$

Passing to a subsequence if necessary, we may assume without loss of generality that

$$\|a_n\|_{\mathscr{E}} < m_n^{-1}2^{-n}, \quad \|\delta(a_n) - a\|_{\mathscr{E}} < m_n^{-1}$$

and that

$$\|a_n\|_{\mathscr{E}} < 2^{-1} n m_n^{-1} \|\delta(e_n)\|_{\mathscr{M}}^{-1}$$

whenever $n \geqslant 1$ is such that $\delta(e_n) \neq 0$. Let us define an element

$$c := \sum_{n=1}^{\infty} m_n e_n a_n e_n \in \mathscr{E}$$

where the series converges in the norm $\|\cdot\|_{\mathscr{E}}$, since we have $\|m_n e_n a_n e_n\|_{\mathscr{E}} < 2^{-n}$. We intend to obtain a contradiction by showing that the norm $\|\delta(c)\|_{\mathscr{E}}$ is larger than any positive integer n.

Indeed, fixing such $n \geqslant 1$, we have $\|\delta(c)\|_{\mathscr{E}} \geqslant \|e_n \delta(c) e_n\|_{\mathscr{E}}$ and

$$\begin{aligned} \|e_n\delta(c)e_n\|_{\mathscr{E}} &= \|\delta(e_n c)e_n - \delta(e_n)ce_n\|_{\mathscr{E}} = m_n\|\delta(e_n a_n e_n)e_n - \delta(e_n)e_n a_n e_n\|_{\mathscr{E}} \\ &= m_n\|e_n\delta(e_n a_n e_n)e_n\|_{\mathscr{E}} = m_n\|e_n\delta(a_n)e_n + e_n\delta(e_n)a_n e_n + e_n a_n\delta(e_n)e_n\|_{\mathscr{E}} \\ &\geqslant m_n\|e_n(\delta(a_n) - a)e_n + e_n a e_n\|_{\mathscr{E}} - m_n\|e_n\delta(e_n)a_n e_n\|_{\mathscr{E}} - m_n\|e_n a_n\delta(e_n)e_n\|_{\mathscr{E}} \\ &\geqslant m_n(\|e_n a e_n\|_{\mathscr{E}} - \|e_n(\delta(a_n) - a)e_n\|_{\mathscr{E}}) - 2m_n\|a_n\|_{\mathscr{E}}\|\delta(e_n)\|_{\mathscr{M}} \\ &\geqslant m_n(\|e_n a e_n\|_{\mathscr{E}} - \|\delta(a_n) - a\|_{\mathscr{E}}) - n > m_n\|e_n a e_n\|_{\mathscr{E}} - 1 - n \geqslant m_n\|e_n\|_{\mathscr{E}} - 1 - n > n. \end{aligned}$$

This shows that δ is a continuous mapping on $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$. $\rhd$

Proposition 3. *Let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a Banach $*$-ideal in the von Neumann algebra $\mathscr{M}$ and let δ be a derivation on $\mathscr{E}$. Then δ is a continuous mapping on $(\mathscr{E}, \|\cdot\|_{\mathscr{M}})$ and $\|\delta\|_\infty := \|\delta\|_{(\mathscr{E},\|\cdot\|_{\mathscr{M}})\to(\mathscr{E},\|\cdot\|_{\mathscr{M}})} \leqslant 2\|\delta\|$, where $\|\delta\| = \|\delta\|_{(\mathscr{E},\|\cdot\|_{\mathscr{E}})\to(\mathscr{E},\|\cdot\|_{\mathscr{E}})}$.*

$\lhd$ By Proposition 2, a derivation $\delta\colon \mathscr{E} \to \mathscr{E}$ is a continuous on $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$, in particular, $\|\delta\| := \|\delta\|_{\mathscr{E}\to\mathscr{E}} < \infty$.

Let $x \in \mathscr{E}$ and $\delta(x) \neq 0$. Let $0 < \varepsilon < \|\delta(x)\|_{\mathscr{M}}$ and denote by p_x the spectral projection of the operator $|\delta(x)|$ corresponding to the segment $[\|\delta(x)\|_{\mathscr{M}} - \varepsilon, \|\delta(x)\|_{\mathscr{M}}]$. Using Gelfand–Naimark theorem, one can obtain that $p_x \neq 0$.

We have that $0 < (\|\delta(x)\|_{\mathscr{M}} - \varepsilon)p_x \leqslant |\delta(x)|p_x$. Then $p_x \in \mathscr{E}$ and

$$|\delta(x)|p_x = (p_x|\delta(x)|^2 p_x)^{1/2} = (p_x\delta(x)^*\delta(x)p_x)^{1/2} = |\delta(x)p_x|.$$

Since the norm $\|\cdot\|_{\mathscr{E}}$ is monotone, we obtain that

$$\begin{aligned} (\|\delta(x)\|_{\mathscr{M}} - \varepsilon)\|p_x\|_{\mathscr{E}} &\leqslant \|\delta(x)p_x\|_{\mathscr{E}} = \|\delta(xp_x) - x\delta(p_x)\|_{\mathscr{E}} \leqslant \|\delta(xp_x)\|_{\mathscr{E}} + \|x\delta(p_x)\|_{\mathscr{E}} \\ &\leqslant \|\delta(xp_x)\|_{\mathscr{E}} + \|x\|_{\mathscr{M}}\|\delta(p_x)\|_{\mathscr{E}} \leqslant \|\delta\|\|xp_x\|_{\mathscr{E}} + \|x\|_{\mathscr{M}}\|\delta\|\|p_x\|_{\mathscr{E}} \\ &\leqslant \|\delta\|\|x\|_{\mathscr{M}}\|p_x\|_{\mathscr{E}} + \|x\|_{\mathscr{M}}\|\delta\|\|p_x\|_{\mathscr{E}} = 2\|\delta\|\|x\|_{\mathscr{M}}\|p_x\|_{\mathscr{E}}, \end{aligned}$$

that is

$$(\|\delta(x)\|_{\mathscr{M}} - \varepsilon)\|p_x\|_{\mathscr{E}} \leqslant 2\|\delta\|\|x\|_{\mathscr{M}}\|p_x\|_{\mathscr{E}}.$$

Dividing by $\|p_x\|_{\mathscr{E}}$ and using arbitrariness of ε, we infer that

$$\|\delta(x)\|_{\mathscr{M}} \leqslant 2\|\delta\|\|x\|_{\mathscr{M}}.$$

Thus the operator δ is bounded with respect to the norm $\|\cdot\|_{\mathscr{M}}$, in addition, $\|\delta\|_\infty \leqslant 2\|\delta\|$. $\rhd$

Now we give a proof of Theorem 1.

$\lhd$ Proof of Theorem 1. Denote by $\widetilde{\mathscr{E}}$ and $\widehat{\mathscr{E}}$ the closure of the ideal $\mathscr{E}$ with respect to the uniform and weak operator topology, respectively. Then $\mathscr{E} \subset \widetilde{\mathscr{E}} \subset \widehat{\mathscr{E}}$. It is clear that $\widetilde{\mathscr{E}}$ is a C^*-subalgebra in $\mathscr{M}$ and the derivation δ extends by continuity (see Proposition 3) up to a derivation $\widetilde{\delta} : \widetilde{\mathscr{E}} \to \widetilde{\mathscr{E}}$, in addition $\|\widetilde{\delta}\|_\infty = \|\delta\|_\infty$.

Since $\widehat{\mathscr{E}}$ is a *wo*-closed two-sided in $\mathscr{M}$, it follows, by Proposition 1, that $\widehat{\mathscr{E}} = z \cdot \mathscr{M}$ for some central projection z in $\mathscr{M}$. Then $\widehat{\mathscr{E}}$ is a W^*-subalgebra with the identity z. By Theorem 2, the derivation $\widetilde{\delta}$ extends up to a derivation $\widehat{\delta} : \widehat{\mathscr{E}} \longrightarrow \widehat{\mathscr{E}}$, in addition, there exists an element $a \in \widehat{\mathscr{E}}$ such that $\delta(x) = \delta_a(x) = [a, x]$ for all $x \in \mathscr{E}$ and $\|a\|_{\mathscr{M}} \leqslant \frac{1}{2}\|\delta_a\|_\infty = \frac{1}{2}\|\delta\|_\infty \leqslant \|\delta\|$. $\rhd$

Corollary 1 (cf. [6, Theorem 3.2])**.** *Let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a Banach ideal of compact operators in $\mathscr{B}(H)$ and let $\delta\colon \mathscr{E} \to \mathscr{E}$ be a derivation on $\mathscr{E}$. Then there exists an operator $a \in \mathscr{B}(H)$ such that $\delta(x) = [a, x]$ for all $x \in \mathscr{E}$. Moreover, we can choose such an element a as follows: $\|a\|_{\mathscr{M}} \leqslant \|\delta\|_{\mathscr{E}\to\mathscr{E}}$.*

Corollary 2 (cf. [12, Theorem 5.2])**.** *Let $\mathscr{M}$ be a commutative von Neumann algebra and let $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a Banach ∗-ideal in $\mathscr{M}$. Then any derivation δ on $\mathscr{E}$ vanishes.*

A detailed study of derivations on the ideals in commutative AW^*-algebras is given in the paper [12]. In particular, it is shown here that if the Boolean algebra $\mathscr{P}(\mathscr{M})$ of all projections in the commutative AW^*-algebra $\mathscr{M}$ is not σ-distributive then there exists a non-zero derivation on ideals in $\mathscr{M}$ with values in a commutative ∗-algebra $C_\infty(Q) \oplus i \cdot C_\infty(Q)$, where Q is a Stone compactum corresponding to the Boolean algebra $\mathscr{P}(\mathscr{M})$. An analogous result for derivations on an algebra $C_\infty(Q, \mathbb{C})$ was earlier obtained by A. G. Kusraev [13] for a general Stone compactum.

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Received March 21, 2018

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Владикавказский математический журнал
2018, Том 20, Выпуск 2, С. 23–28

ДИФФЕРЕНЦИРОВАНИЯ В БАНАХОВЫХ $*$-ИДЕАЛАХ АЛГЕБР ФОН НЕЙМАНА

Бер А. Ф., Чилин В. И., Сукочев Ф. А.

Аннотация. Известно, что любое дифференцирование $\delta : M \to M$ на алгебре фон Неймана $\mathscr{M}$ является внутренним, т. е. $\delta(x) := \delta_a(x) = [a, x] = ax - xa$, $x \in \mathscr{M}$, для некоторого $a \in \mathscr{M}$. Если H сепарабельное бесконечномерное гильбертово пространство и $\mathscr{K}(H)$ есть C^*-подалгебра компактных операторов в C^*-алгебре $\mathscr{B}(H)$ всех ограниченных линейных операторов, действующих в H, то каждое дифференцирование $\delta : \mathscr{K}(H) \to \mathscr{K}(H)$ есть специальное дифференцирование, т. е. существует такой оператор $a \in \mathscr{B}(H)$, что $\delta(x) = [x, a]$ для всех $x \in K(H)$. В недавней работе А. Ф. Бера, В. И. Чилина, Г. Б. Левитиной, Ф. А. Сукочева (JMAA, 2013) установлено, что каждое дифференцирование $\delta\colon \mathscr{E} \to \mathscr{E}$ на любом банаховом симметричном идеале компактных операторов $\mathscr{E} \subseteq \mathscr{K}(H)$ также является пространственным. Мы показываем, что аналогичный результат верен и для произвольных банаховых $*$-идеалов в любой алгебре фон Неймана $\mathscr{M}$. Более точно: Если $\mathscr{M}$ любая алгебра фон Неймана, $\mathscr{E}$ банаховый $*$-идеал в $\mathscr{M}$ и $\delta\colon \mathscr{E} \to \mathscr{E}$ есть дифференцирование на $\mathscr{E}$, то существует такой элемент $a \in \mathscr{M}$, что $\delta(x) = [x, a]$ для всех $x \in \mathscr{E}$, т. е. δ есть пространственное дифференцирование.

Ключевые слова: алгебра фон Неймана, банахов $*$-идеал, дифференцирование, пространственное дифференцирование.