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UNBOUNDED CONVERGENCE IN THE CONVERGENCE VECTOR LATTICES: A SURVEY

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Dedicated to Professor A. G. Kusraev
on the occasion of his 65th anniversary

Various convergences in vector lattices were historically a subject of deep investigation which stems from the begining of the 20th century in works of Riesz, Kantorovich, Nakano, Vulikh, Zanen, and many other mathematicians. The study of the unbounded order convergence had been initiated by Nakano in late 40th in connection with Birkhoff's ergodic theorem. The idea of Nakano was to define the almost everywhere convergence in terms of lattice operations without the direct use of measure theory. Many years later it was recognised that the unbounded order convergence is also rathe useful in probability theory. Since then, the idea of investigating of convergences by using their unbounded versions, have been exploited in several papers. For instance, unbounded convergences in vector lattices have attracted attention of many researchers in order to find new approaches to various problems of functional analysis, operator theory, variational calculus, theory of risk measures in mathematical finance, stochastic processes, etc. Some of those unbounded convergences, like unbounded norm convergence, unbounded multi-norm convergence, unbounded τ-convergence are topological. Others are not topological in general, for example: the unbounded order convergence, the unbounded relative uniform convergence, various unbounded convergences in lattice-normed lattices, etc. Topological convergences are, as usual, more flexible for an investigation due to the compactness arguments, etc. The non-topological convergences are more complicated in genelal, as it can be seen on an example of the a.e-convergence. In the present paper we present recent developments in convergence vector lattices with emphasis on related unbounded convergences. Special attention is paid to the case of convergence in lattice multi pseudo normed vector lattices that generalizes most of cases which were discussed in the literature in the last 5 years.

Key words: convergence vector lattice, lattice normed lattice, unbounded convergence.
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1. Introduction

A *convergence* [*s-convergence*] $\mathbb{c}$ for nets [resp., for sequences] in a set X is defined by the following two conditions:

(a) $x_\alpha \equiv x \Rightarrow x_\alpha \xrightarrow{\mathbb{c}} x$ [resp., $x_n \equiv x \Rightarrow x_n \xrightarrow{\mathbb{c}} x$];

(b) $x_\alpha \xrightarrow{\mathbb{c}} x \Rightarrow x_\beta \xrightarrow{\mathbb{c}} x$ for every subnet x_β of x_α [resp., $x_n \xrightarrow{\mathbb{c}} x \Rightarrow x_{n_k} \xrightarrow{\mathbb{c}} x$ for every subsequence x_{n_k} of x_n].

A *convergence set* is a pair $(X, \mathbb{c})$ where $\mathbb{c}$ is a convergence in a set X. A mapping f from a convergence set $(X_1, \mathbb{c}_1)$ into a convergence set $(X_2, \mathbb{c}_2)$ is said to be *continuous*, if $x_\alpha \xrightarrow{\mathbb{c}_1} x$ implies $f(x_\alpha) \xrightarrow{\mathbb{c}_2} f(x)$. s-Continuity of f is defined by replacing nets with sequences.

A subset A of $(X,\mathbb{c})$ is called: $\mathbb{c}$-*closed* if $A \ni x_\alpha \xrightarrow{\mathbb{c}} x \Rightarrow x \in A$; $\mathbb{c}$-*compact* if any net a_α in A possesses a subnet a_β such that $a_\beta \xrightarrow{\mathbb{c}} a$ for some $a \in A$. $\mathbb{s}\mathbb{c}$-*Closedness* and $\mathbb{s}\mathbb{c}$-*compactness* are defined by using sequences. If the set $\{x\}$ is $\mathbb{c}$-closed for every $x \in X$ then $\mathbb{c}$ is called T_1-*convergence*. It is immediate to see that $\mathbb{c} \in T_1$ if every constant net $x_\alpha \equiv x$ does not $\mathbb{c}$-converge to any $y \neq x$. For further information on convergences we refer to [1, 2].

In the present paper, we investigate several special convergences in *real vector lattices*. Under *convergence* in a vector lattice X we always understand a convergence $\mathbb{c}$ in the set X, which agrees with the linear and lattice operations in the following way:

$$X \ni x_\alpha \xrightarrow{\mathbb{c}} x, \quad X \ni y_\beta \xrightarrow{\mathbb{c}} y, \quad \mathbb{R} \ni r_\gamma \to r$$

imply

$$r_\gamma \cdot x_\alpha + y_\beta \xrightarrow{\mathbb{c}} r \cdot x + y \tag{1}$$

and

$$r_\gamma \cdot x_\alpha \wedge y_\beta \xrightarrow{\mathbb{c}} r \cdot x \wedge y. \tag{2}$$

In other words, the linear and lattice operations in X are continuous with respect to the $\mathbb{c}$-convergence in X and to the usual convergence in $\mathbb{R}$. In this case, we say that $X = (X,\mathbb{c})$ is a *convergence vector lattice*. $\mathbb{s}$-*Convergence vector lattices* are defined by using in (1) and (2) sequences instead of nets.

A net x_α [resp., a sequence x_n] in $(X,\mathbb{c})$ is called a $\mathbb{c}$-*Cauchy*, whenever

$$(x_\alpha - x_\beta) \xrightarrow{\mathbb{c}} 0 \quad [\text{resp., } (x_m - x_n) \xrightarrow{\mathbb{c}} 0 \quad (m,n \to \infty)]. \tag{3}$$

A convergence vector lattice $(X,\mathbb{c})$ is said to be $\mathbb{c}$-*complete* [resp., $\mathbb{s}\mathbb{c}$-*complete*], if every $\mathbb{c}$-Cauchy net [resp., $\mathbb{c}$-Cauchy sequence] in X is $\mathbb{c}$-convergent.

A T_1-convergence $\mathbb{c}_1$ in a vector lattice X is said to be *minimal* [$\mathbb{s}$-*minimal*], if for any other T_1-convergence $\mathbb{c}$ in X satisfying $x_\alpha \xrightarrow{\mathbb{c}_1} 0 \Rightarrow x_\alpha \xrightarrow{\mathbb{c}} 0$ for all nets x_α in X [resp., $x_n \xrightarrow{\mathbb{c}_1} 0 \Rightarrow x_n \xrightarrow{\mathbb{c}} 0$ for all sequences x_n in X], it follows that $\mathbb{c} = \mathbb{c}_1$.

A convergence $\mathbb{c}$ in a vector lattice X is said to be *Lebesgue* [resp., $\mathbb{s}$-*Lebesgue*], if for every net x_α [resp., for every sequence x_n] in X

$$x_\alpha \xrightarrow{o} 0 \Longrightarrow x_\alpha \xrightarrow{\mathbb{c}} 0, \tag{4}$$

[respectively,

$$x_n \xrightarrow{o} 0 \Longrightarrow x_n \xrightarrow{\mathbb{c}} 0]. \tag{5}$$

It follows from (4), (5) that every Lebesgue convergence is $\mathbb{s}$-Lebesgue.

Basic examples of convergence vector lattices are: a locally solid vector lattice $X = (X,\tau)$ with its τ-*convergence* [3]; a space of Lebesgue measurable functions on $[0,1]$ with the *almost everywhere convergence*, that is a $\mathbb{s}\mathbb{c}$-Lebesgue convergence; a vector lattice X with the $\mathbb{o}$-*convergence* [$\mathbb{r}\mathbb{u}$-*convergence*] [4]; a lattice normed vector lattice (X,p,E) with the $\mathbb{p}$-*convergence* [5, 6]. For more details, see [3–10].Recently, $\mathbb{o}$- and $\mathbb{u}\mathbb{o}$-convergence were investigated in [7; 11–16] with some further applications in [17–19].

In the present paper, we introduce several further convergence lattices and investigate corresponding unbounded convergences.

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2. Examples of convergence vector lattices

In this section, we collect and shortly discuss several examples of convergence vector lattices. The convergences in Examples 2, 3, and 4 below are topological in the sense that there is locally solid topology τ such that the τ-convergence coincided with the corresponding $\mathbb{c}$-convergence.

EXAMPLE 1. Let X be a vector lattice. Clearly, $(X, \xrightarrow{o})$ is a T_1-convergence vector lattice. Furthermore, $(X, \xrightarrow{\text{ru}})$ is a convergence vector lattice, where "$\xrightarrow{o}$" is T_1 iff X is Archimedean (cf. [3, 4, 8, 9, 10]).

In a Lebesgue and complete metrizable locally solid vector lattice, $x_\alpha \xrightarrow{\text{ru}} x$ iff $x_\alpha \xrightarrow{o} x$ [20, Proposition 3]. It was also shown in [20, Proposition 4] that, in $\mathbb{R}^\Omega$, "$\xrightarrow{\text{ru}}$" is equivalent to "$\xrightarrow{o}$" for nets iff Ω is countable. Furthermore, it was proved that the $\mathbb{o}$-convergence in X is topological iff $\dim(X) < \infty$ [11, Theorem 1], and that the $\mathbb{RU}$-convergence is topological iff X has a strong order unit [20, Theorem 5]. It is worth to notice that the *so*-convergence in a Banach lattice X of countable type coincides with the norm convergence iff X is lattice isomorphic to c_0 [21, Theorem 1].

EXAMPLE 2. Let $\mathscr{M} = \{m_\xi\}_{\xi\in\Xi}$ be a family of Riesz seminorms on a vector lattice X. If, for any $0 \neq x \in X$, there is $m_\xi \in \mathscr{M}$ such that $m_\xi(x) > 0$, $(X, \mathscr{M})$ is said to be a *multi-normed lattice* (cf. [10, Definition 5.1.6]), abbreviated by *MNL*, with the *Riesz multi-norm* $\mathscr{M}$. Convergence in a Riesz multi-norm ($\mathbb{M}$-*convergence*) was studied recently in [7].

MNLs are also known as Hausdorff locally convex-solid vector lattices (cf. [3, p. 59]). Note that now-days the name "multi-normed space" is also used for quite different class of spaces [22].

EXAMPLE 3. Given a vector lattice X, a function $r : X \to \mathbb{R}_+$ is called a *Riesz pseudoseminorm* (cf. [3, Definition 2.27]), whenever:

(*a*) $r(x + y) \leqslant r(x) + r(y)$ for all $x, y \in X$;
(*b*) $\lim_{n\to\infty} r(\alpha_n x) = 0$ for all $x \in X$ and for all $\mathbb{R} \ni \alpha_n \to 0$;
(*c*) $|y| \geqslant |x|$ implies $r(y) \geqslant r(x)$.

If $r(x) \neq 0$ for any $0 \neq x \in X$, r is called a *Riesz pseudonorm* and (X, r) is said to be a *pseudonormed lattice* (abbreviated by *PNL*).

The convergence in a PNL is rather similar to the norm convergence in a normed lattice except of possible lack of a locally convex base for the corresponding topology.

The next example presents a convergence which generalizes convergences from Examples 2 and 3.

EXAMPLE 4. We say that a collection $\mathscr{R} = \{r_\xi\}_{\xi\in\Xi}$ of Riesz pseudoseminorms on X is a *Riesz multi-pseudonorm*, if for any $0 \neq x \in X$, there is $r_\xi \in \mathscr{R}$ with $r_\xi(x) > 0$. In this case, $(X, \mathscr{R})$ is said to be a *multi-pseudonormed lattice* (abbreviated by *MPNL*).

Notice that, by the Fremlin theorem (cf. [3, Theorem 2.28]), MPNLs are exactly the locally solid vector lattices.

The *Riesz multi-pseudonorm convergence* ($\mathbb{MP}$-*convergence*) in $(X, \mathscr{R})$,

$$x_\alpha \xrightarrow{\text{mp}} x \iff (\forall\, r_\xi \in \mathscr{R})\ \ r_\xi(x - x_\alpha) \to 0, \tag{6}$$

coincides with τ-convergence, where τ is the corresponding locally solid topology in $(X, \mathscr{R})$.

EXAMPLE 5. Given vector lattices X and E, a function $p : X \to E_+$ is called an E-valued Riesz seminorm (cf. [4, 9]), whenever:

(*a*) $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$;
(*b*) $p(\alpha x) = |\alpha| \cdot p(x)$ for all $x \in X$, $\alpha \in \mathbb{R}$;

(c) $|y| \geqslant |x|$ implies $p(y) \geqslant p(x)$.

If, additionally, $p(x) \neq 0$ for any $0 \neq x \in X$, we say that p is an *E-valued Riesz norm.*

A vector lattice (X, p, E) equipped with an E-valued Riesz norm p is called a *lattice normed lattice* (abbreviated by *LNL*).

Several types of convergences in lattice normed lattices were studied recently in [5, 6, 23]. One of the most interesting convergences here is the $\mathbb{p}$-*convergence*:

$$x_\alpha \xrightarrow{\mathrm{p}} x \iff p(x - x_\alpha) \xrightarrow{o} 0. \tag{7}$$

Notice that, the $\mathbb{p}$-convergence in $(X, |\cdot|, X)$ coincides with the $\mathbb{o}$-convergence in X which is not topological if $\dim(X) = \infty$.

Example 6. A vector lattice $X = (X, \mathscr{M}, E)$ equipped with a separating family $\mathscr{M} = \{p_\xi\}_{\xi \in \Xi}$ of *E-valued Riesz seminorms* is said to be a *lattice multi-normed lattice* (abbreviated by *LMNL*). The corresponding convergence:

$$x_\alpha \xrightarrow{\mathrm{lm}} x \iff (\forall\, p_\xi \in \mathscr{M})\ \ p_\xi(x - x_\alpha) \xrightarrow{o} 0 \tag{8}$$

is called the $\mathbb{lm}$-*convergence.* Clearly, any LNL is an LMNL.

Example 7. Given two vector lattices X and E. A function $\rho : X \to E_+$ is called an *E-valued Riesz pseudonorm*, whenever:

(a) $\rho(x + y) \leqslant \rho(x) + \rho(y)$ for all $x, y \in X$;

(b) $\rho(\alpha_n x) \xrightarrow{o} 0$ for all $x \in X$ and $\mathbb{R} \ni \alpha_n \to 0$;

(c) $|y| \geqslant |x|$ implies $\rho(y) \geqslant \rho(x)$;

(d) $x \neq 0$ implies $\rho(x) \neq 0$.

If condition (d) is dropped, ρ is said to be an *E-valued Riesz pseudoseminorm.*

A vector lattice X equipped with an E-valued Riesz pseudonorm ρ is called a *lattice pseudonormed lattice* (abbreviated by *LPNL* and denoted by (X, ρ, E)). The corresponding convergence:

$$x_\alpha \xrightarrow{\mathrm{p}} x \iff p(x - x_\alpha) \xrightarrow{o} 0 \tag{9}$$

is called, as in Example 5, the $\mathbb{p}$-*convergence* in (X, ρ, E). Clearly, any LNL is an LPNL.

Our last example presents a convergence which generalizes convergences from all previous examples except the $\mathbb{ru}$-convergence from Example 1.

Example 8. A family $\mathscr{R} = \{\rho_\xi\}_{\xi \in \Xi}$ of E-valued Riesz pseudoseminorms is said to be *separating* whenever, for any $0 \neq x \in X$, there is $\rho_\xi \in \mathscr{R}$ such that $\rho_\xi(x) > 0$. If $\mathscr{R}$ is separating, we call it an *E-valued Riesz multi-pseudonorm.*

A vector lattice $(X, \mathscr{R}, E)$ equipped with an *E-valued Riesz multi-pseudonorm* $\mathscr{R}$ is said to be a *lattice multi-pseudonormed lattice* (abbreviated by *LMPNL*). The corresponding convergence:

$$x_\alpha \xrightarrow{\mathrm{lmp}} x \iff (\forall\, \rho_\xi \in \mathscr{R})\ \ \rho_\xi(x - x_\alpha) \xrightarrow{o} 0 \tag{10}$$

is called the $\mathbb{lmp}$-*convergence.*

3. Unbounded convergences

Various unbounded convergences have been investigated recently in [5, 7, 11–16, 18, 20, 24–29, 30–32]. This section is focused on the unification of approaches for unbounded convergences in different settings. After this, we discuss several types of unbounded convergences related to examples in Section 2.

3.1. General facts. Let I be an ideal in a convergence vector lattice $(X, \mathbb{c})$. The following definition is motivated by the definition of *un-convergence with respect to an ideal I of a normed lattice* $(X, \|\cdot\|)$ [14].

Definition 1. The *unbounded $\mathbb{c}$-convergence w. r. to I* (shortly, *$\mathbb{u}_I\mathbb{c}$-convergence*) is defined by

$$x_\alpha \xrightarrow{\mathrm{u_I c}} x \text{ if } |x_\alpha - x| \wedge u \xrightarrow{\mathbb{c}} x \quad \text{for all} \quad u \in I_+. \tag{11}$$

It follows directly from (1) and (2), that $(X, \mathbb{u}_I\mathbb{c})$ is a convergence vector lattice, where

$$\mathbb{u}_I\mathbb{c} \in T_1 \iff \mathbb{c} \in T_1 \text{ and } I \text{ is order dense.}$$

Furthermore, the $\mathbb{u}_I\mathbb{c}$-convergence is coarser than $\mathbb{c}$ and $\mathbb{u}_I\mathbb{u}_I\mathbb{c} = \mathbb{u}_I\mathbb{c}$. Thus, if I is order dense and $\mathbb{c}$ is T_1 and minimal, then $\mathbb{u}_I\mathbb{c} = \mathbb{c}$. If $\mathbb{c}$ is topological, then $\mathbb{u}_I\mathbb{c}$ is topological as well (cf. [20, 30]). Unbounded $\mathbb{c}$-convergence w. r. to $I = X$ is denoted by $\mathbb{uc}$.

The $\mathbb{uo}$-convergence was studied recently in [11, 13, 16, 18, 27, 28, 31]. The $\mathbb{un}$-convergence was introduced and investigated in [12] (see also [14, 15, 29, 32]). We refer to [5, 6] for the $\mathbb{up}$-convergence; to [7] for the $\mathbb{um}$-convergence; and to [20, 29, 30, 31, 33] for the $\mathbb{u}\tau$-convergence.

It may happened that a $\mathbb{c}$-convergence is not topological, yet the $\mathbb{uc}$-convergence is topological. For example, if X is an atomic order continuous Banach lattice, then the $\mathbb{uo}$-convergence in X is topological [12, Theorem 5.3], whereas the $\mathbb{o}$-convergence in X is not topological except $\dim(X) < \infty$ [11, Theorem 1].

The following proposition is a $\mathbb{uc}$-version of [13, Proposition 3.15] (cf. also [5, Proposition 3.11] and [30, Proposition 2.12]). Since its proof is similar, we omit it.

Proposition 1. *Let $\mathbb{c}$ be a Lebesgue T_1-convergence in a vector lattice X and Y a sublattice of X. Y is $\mathbb{uc}$-closed iff it is $\mathbb{c}$-closed.*

It was shown in [30, Theorem 6.4] that in a Hausdorff locally solid vector lattice (X, τ) the τ-convergence minimal iff it is Lebesgue and $\mathbb{u}\tau = \tau$. The question, whether or not any T_1-convergence in a vector lattice is minimal iff it is Lebesgue and $\mathbb{uc} = \mathbb{c}$, remains open.

Two further questions arise in the case of topological $\mathbb{uc}$-convergence (i. e. $\mathbb{uc}$ is a τ-convergence for some locally solid τ in X). Under which conditions the topology τ is locally convex? Metrizable? In the case of $\mathbb{n}$-convergence (norm convergence) in a Banach lattice X, it was proved that: (1) $\mathbb{un}$-topology is metrizable iff X has a quasi-interior point [15, Theorem 3.2]; (2) if X is order continuous, then $(X, \mathbb{un})$ is locally convex iff X is atomic [15, Theorem 5.2]. In the general case, no investigation was conducted yet.

3.2. $\mathbb{uo}$-Convergence and $\mathbb{uru}$-convergence. The $\mathbb{uo}$-convergence was studied deeply in many recent papers (cf. [11–14, 26–28, 31]), whereas the $\mathbb{uru}$-convergence was investigated in [5, 7, 11, 20]. It was proved [20, Proposition 3] that in a Lebesgue and complete metrizable locally solid vector lattice X, $x_\alpha \xrightarrow{\mathrm{ru}} x \iff x_\alpha \xrightarrow{o} x$ for every net x_α. In [20, Proposition 4], it was shown that, in $X = \mathbb{R}^\Omega$, "$\xrightarrow{\mathrm{ru}}$" is equivalent to "$\xrightarrow{o}$" for nets iff Ω is countable. Furthermore, it was proved in [11], that the $\mathbb{o}$-convergence is topological iff $\dim(X) < \infty$ [11, Theorem 1], and that the $\mathbb{ru}$-convergence is topological iff X has a strong order unit [11, Theorem 5].

3.3. $\mathbb{um}$-Convergence and $\mathbb{u}\tau$-convergence. Recently, $\mathbb{um}$-Convergence was studied in [7], whereas $\mathbb{u}\tau$-convergence in [7, 29, 30, 33]. Among other things, it was shown that in a metrizable $\mathbb{m}$-complete MNL $(X, \mathscr{M})$ the $\mathbb{um}$-convergence is metrizable iff X has a quasi-interior point [7, Proposition 4]. In [20, Proposition 5] it was shown that in a complete metrizable locally solid vector lattice (X, τ) with a countable topological orthogonal system, the $\mathbb{u}\tau$-convergence is metrizable.

Notice that, in the case of the $\mathbb{m}$-convergence in an MNL $(X, \mathscr{M})$ with the Riesz multi-norm $\mathscr{M} = \{m_\xi\}_{\xi\in\Xi}$, the $\mathbb{um}$-convergence in X is the $\mathbb{mp}$-convergence in the MPNL $(X, \mathscr{R})$, where $\mathscr{R} = \{m_{\xi,u}\}_{\xi\in\Xi, u\in X_+}$ is given by

$$m_{\xi,u}(x) = m_\xi(|x| \wedge u) \quad (\xi \in \Xi,\, u \in X_+). \tag{12}$$

In the case of a locally solid vector lattice (X, τ), in order to describe the $\mathbb{u}\tau$-convergence, we consider a Riesz multi-pseudonorm on X, say $\mathscr{P} = \{\rho_\xi\}_{\xi\in\Xi}$, generating topology τ (such a Riesz multi-pseudonorm exists by the Fremlin theorem). Now, the $\mathbb{u}\tau$-convergence in X is the $\mathbb{mp}$-convergence in the MPNL $(X, \mathscr{R})$, where $\mathscr{R} = \{\rho_{\xi,u}\}_{\xi\in\Xi, u\in X_+}$ is given by:

$$\rho_{\xi,u}(x) = \rho_\xi(|x| \wedge u) \quad (\xi \in \Xi,\, u \in X_+). \tag{13}$$

3.4. Unbounded $\mathbb{p}$-, $\mathbb{lm}$-, and $\mathbb{lmp}$-convergences. The $\mathbb{up}$-convergence was introduced and investigated in [5]. As in (12) above, it can be seen that the $\mathbb{up}$-convergence in X is the $\mathbb{lmp}$-convergence in the LMPNL $(X, \mathscr{P}, E)$, where $\mathscr{P} = \{\pi_u\}_{u\in X_+}$ is given by

$$\pi_u(x) = p(|x| \wedge u) \quad (u \in X_+). \tag{14}$$

In the case of an LMNL $X = (X, \mathscr{M}, E)$ with the E-valued Riesz multi-norm $\mathscr{M} = \{p_\xi\}_{\xi\in\Xi}$, the $\mathbb{ulm}$-convergence in X is the $\mathbb{lmp}$-convergence in the LMPNL $(X, \mathscr{P}, E)$, where $\mathscr{P} = \{\pi_{\xi,u}\}_{\xi\in\Xi, u\in X_+}$ consists of E-valued Riesz pseudoseminorms $\pi_{\xi,u}$ defined by

$$\pi_{\xi,u}(x) = p_\xi(|x| \wedge u) \quad (x \in X). \tag{15}$$

Furthermore, in the most general case of the $\mathbb{lmp}$-convergence from Example 8, we have the following proposition, whose straightforward proof is omitted.

Proposition 2. *Let $X = (X, \mathscr{R}, E)$ be an LMPNL with the E-valued Riesz multi-pseudonorm $\mathscr{R} = \{\rho_\xi\}_{\xi\in\Xi}$. Then the $\mathbb{ulmp}$-convergence in X is the $\mathbb{lmp}$-convergence in the LMPNL $(X, \mathscr{P}, E)$, where $\mathscr{P}$ consists of E-valued Riesz pseudoseminorms $\pi_{\xi,u}$*

$$\pi_{\xi,u}(x) = \rho_\xi(|x| \wedge u) \quad (x \in X) \tag{16}$$

for all $\xi \in \Xi, u \in X_+$.

For more results on $\mathbb{up}$-convergence we refer to [5, 6, 24, 25].

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НЕОГРАНИЧЕННЫЕ СХОДИМОСТИ В КОНВЕРГЕНТНЫХ ВЕКТОРНЫХ РЕШЕТКАХ

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Исторически, разнообразные сходимости в векторных решетках являлись предметом глубоких исследований восходящих к началу XX века. Изучение неограниченной порядковой сходимости было инициировано Накано в конце 40-х годов, в связи с эргодической теоремой Биркгофа. Идея Накано заключалась в том, чтобы определить сходимость почти всюду в терминах решеточных операций без прямого использования теории меры. Много лет спустя выяснилось, что неограниченная порядковая сходимость весьма полезна в теории вероятностей. С тех пор идея исследования различных сходимостей с помощью их неограниченных версий используется в различных контекстах. Например, неограниченные сходимости в векторных решетках привлекли внимание многих исследователей для того чтобы найти новые подходы к различным проблемам функционального анализа, теории операторов, вариационного исчисления, теории рисков в финансовой математике и т. д. Некоторые неограниченные сходимости, такие как неограниченная сходимость по норме или мультинорме, неограниченная τ-сходимость, являются топологическими. Другие приведенные сходимости не являются топологическими в общем случае, например: неограниченная порядковая сходимость, неограниченная относительная равномерная сходимость, различные неограниченные сходимости в решеточно-нормированных решетках, и т. п. В настоящей работе представлены последние наиболее часто используемые сходимости в векторных решетках, с акцентом на соответствующих неограниченных сходимостях. Особое внимание уделяется случаю сходимости в решеточно мультипсевдонормированных векторных решетках, обобщающих большинство случаев, обсуждавшихся в литературе за последние 5 лет.

Ключевые слова: конвергентная векторная решетка, решеточно-нормированное пространство, неограниченная сходимость.