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ASYMPTOTIC ALMOST AUTOMORPHY FOR ALGEBRAS OF GENERALIZED FUNCTIONS

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Abstract. The paper aims to to study the concept of asymptotic almost automorphy in the context of generalized functions. We introduce an algebra of asymptotically almost automorphic generalized functions which contains the space of smooth asymptotically almost automorphic functions as a subalgebra. The fundamental importance of this algebra, is related to the impossibility of multiplication of distributions; it also contains the asymptotically almost automorphic Sobolev–Schwartz distributions as a subspace. Moreover, it is shown that the introduced algebra is stable under some nonlinear operations. As a by pass result, the paper gives a Seeley type result on extension of functions in the context of the algebra of bounded generalized functions and the algebra of bounded generalized functions vanishing at infinity, these results are used to prove the fundamental result on the uniqueness of decomposition of an asymptotically almost automorphic generalized function. As applications, neutral difference-differential systems are considered in the framework of the algebra of generalized functions.

Keywords: asymptotic almost automorphy, generalized functions, neutral difference differential equations.

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1. Introduction

Bochner S. defined explicitly almost automorphic functions in the papers [1, 2], where also some of their basic properties are given. In [3] he studied linear difference differential equations in the framework of such functions. It is well known that the concept of almost automorphy is strictly more general than the almost periodicity of H. Bohr [4], however the Stepanoff almost periodicity [5] and the Levitan almost periodicity [6] don't enter into the Bochner concept. Asymptotic almost periodicity of functions as a perturbation of almost periodic functions by functions vanishing at infinity is due to M. Fréchet in [7]. Asymptotic almost automorphy of classical functions is considered in [8], see also [9]. The almost periodicity and the asymptotic almost periodicity of Sobolev–Schwartz distributions, [10] and [11], are respectively considered by L. Schwartz in [11] and I. Cioranescu in [12], while almost automorphy and asymptotic almost automorphy in the setting of these distributions are respectively the subject of the recent works [9] and [13].

In view of the result [14] on the impossibility of the multiplication of distributions, algebras of generalized functions containing spaces of Sobolev–Schwartz type distributions have been studied, see [15–17]. The concepts of almost periodicity and asymptotic almost periodicity as well as almost automorphy in the context of such algebras of generalized functions are introduced, studied and applied in the papers [18–22]. So, the paper first introduces and studies a class of asymptotically almost automorphic generalized functions, denoted by $\mathscr{G}_{aaa}$. In the sense of multiplication, not only $\mathscr{G}_{aaa}$ is stable under multiplication and it contains the space of asymptotically almost automorphic distributions of [9], but moreover some nonlinear operations are performed within the algebra $\mathscr{G}_{aaa}$. As a by pass result, we give a Seeley type result on extension of functions in the context of the introduced generalized functions, this is needed in the proof of a fundamental result on the uniqueness of decomposition of an asymptotically almost automorphic generalized function. The papers [20] and [21] can be considered as consequences of this work. The paper aims also, as in [7], to lift a Frechet existence result of asymptotically almost automorphic solutions of differential equations to the level of neutral difference differential systems in the framework of $\mathscr{G}_{aaa}$.

It is worth noting that the meaning of generalized functions is utilized differently by authors as distributions or ultradistributions, even as hyperfunctions, but in this work by generalized functions we mean in the sense of the works [15–17].

The paper is organized as follows: section two recalls definitions and some properties of asymptotically almost automorphic functions and asymptotically almost automorphic distributions as in [9]. Section three introduces asymptotically almost automorphic generalized functions and gives some of their important properties. The study of a Seeley type result on extensions of generalized functions is given in section four. In section six nonlinear operations on asymptotically almost automorphic generalized function are studied. The last section is dedicated to linear neutral difference differential systems in the framework of asymptotically almost automorphic generalized functions.

2. Asymptotic Almost Automorphy of Functions and Distributions

Let $\mathscr{C}_b$ denotes the space of bounded and continuous complex-valued functions defined on $\mathbb{R}$, endowed with the norm $\|\cdot\|_{L^\infty(\mathbb{R})}$ of uniform convergence on $\mathbb{R}$, it is well-known that $\big(\mathscr{C}_b, \|\cdot\|_{L^\infty(\mathbb{R})}\big)$ is a Banach algebra.

A complex-valued function g defined and continuous on $\mathbb{R}$ is called almost automorphic if for any sequence $(s_m)_{m\in\mathbb{N}} \subset \mathbb{R}$, one can extract a subsequence $(s_{m_k})_k$ such that

$$\tilde{g}(x) := \lim_{k\to+\infty} g(x + s_{m_k}) \quad (\forall\, x \in \mathbb{R}),$$

and

$$\lim_{k\to+\infty} \tilde{g}(x - s_{m_k}) = g(x) \quad (\forall\, x \in \mathbb{R}).$$

The space of almost automorphic functions on $\mathbb{R}$ is denoted by $\mathscr{C}_{aa}$.

The space $\mathscr{C}_{+,0}$ is the set of all bounded and continuous complex-valued functions defined on $\mathbb{R}$ and vanishing at $+\infty$.

Definition 1. We say that a function $f \in \mathscr{C}_b$ is *asymptotically almost automorphic*, if there exist $g \in \mathscr{C}_{aa}$ and $h \in \mathscr{C}_{+,0}$ such that $f = g + h$ on $\mathbb{J} := [0, +\infty[$. The space of asymptotically almost automorphic functions is denoted by $\mathscr{C}_{aaa}$.

For a study of asymptotically almost automorphic functions and asymptotically almost automorphic distributions see [9] and [13] and the references list therein.

Proposition 1. *The decomposition of an asymptotically almost automorphic function is unique on* $\mathbb{J}$.

Notation 1. If $f \in \mathscr{C}_{aaa}$ and $f = g + h$ on $\mathbb{J}$, where $g \in \mathscr{C}_{aa}$ and $h \in \mathscr{C}_{+,0}$. Due to the uniqueness of the decomposition of f, the function g is said the principal term of f and the function h the corrective term of f, we denote them respectively by f_{aa} and f_{cor}. The notation $f = (f_{aa} + f_{cor}) \in \mathscr{C}_{aaa}$ means that $f_{aa} \in \mathscr{C}_{aa}$, $f_{cor} \in \mathscr{C}_{+,0}$ and $f = f_{aa} + f_{cor}$ on $\mathbb{J}$.

Let $\mathscr{E}(\mathbb{I})$ be the algebra of smooth functions on $\mathbb{I} = \mathbb{R}$ or $\mathbb{J}$, and define the space

$$\mathscr{D}_{L^p}(\mathbb{I}) := \left\{\varphi \in \mathscr{E}(\mathbb{I}) : \forall\, j \in \mathbb{Z}_+,\, \varphi^{(j)} \in L^p(\mathbb{I})\right\}, \quad p \in [1, +\infty],$$

that we endow with the topology defined by the family of semi-norms

$$|\varphi|_{k,p,\mathbb{I}} := \sum_{j \leqslant k} \left\|\varphi^{(j)}\right\|_{L^p(\mathbb{I})}, \quad k \in \mathbb{Z}_+.$$

So, $\mathscr{D}_{L^p}(\mathbb{I})$ is a Fréchet subalgebra of $\mathscr{E}(\mathbb{I})$. Denote $\mathscr{B}(\mathbb{I}) := \mathscr{D}_{L^\infty}(\mathbb{I})$.

Remark 1. We have $\lim_{x \underset{>}{\to} 0} \varphi^{(j)}(x)$ exists for every $j \in \mathbb{Z}_+$ when $\varphi \in \mathscr{D}_{L^p}(\mathbb{J})$.

The space of smooth almost automorphic functions $\mathscr{B}_{aa}$ and smooth asymptotically almost automorphic functions $\mathscr{B}_{aaa}$ are defined respectively by

$$\mathscr{B}_{aa} := \left\{\varphi \in \mathscr{E}(\mathbb{R}) : \forall\, j \in \mathbb{Z}_+,\, \varphi^{(j)} \in \mathscr{C}_{aa}\right\}.$$

$$\mathscr{B}_{aaa} := \left\{\varphi \in \mathscr{E}(\mathbb{R}) : \forall j \in \mathbb{Z}_+,\, \varphi^{(j)} \in \mathscr{C}_{aaa}\right\}.$$

We endow $\mathscr{B}_{aa}$ and $\mathscr{B}_{aaa}$ with the topology induced by $\mathscr{B} := \mathscr{D}_{L^\infty}(\mathbb{R})$.

Proposition 2. (1) *The space $\mathscr{B}_{aaa}$ is a Fréchet subalgebra of $\mathscr{B}$ stable by translation.*
(2) $\mathscr{B}_{aaa} \times \mathscr{B}_{aa} \subset \mathscr{B}_{aaa}$.
(3) $\mathscr{B}_{aaa} * L^1 \subset \mathscr{B}_{aaa}$.

The space of L^p-distributions, $p \in]1, +\infty]$, denoted by $\mathscr{D}'_{L^p}(\mathbb{R})$, is the topological dual of $\mathscr{D}_{L^q}(\mathbb{R})$, where $1/p + 1/q = 1$. Let $\dot{\mathscr{B}}$ be the closure in $\mathscr{B}$ of the space $\mathscr{D} \subset \mathscr{E}(\mathbb{R})$ of functions with compact support. The topological dual of $\dot{\mathscr{B}}$ is denoted by $\mathscr{D}'_{L^1}(\mathbb{R})$. The space of bounded distributions $\mathscr{D}'_{L^\infty}(\mathbb{R})$ is denoted by $\mathscr{B}'$. The following definition follows from the characterizations of almost automorphic distributions, see [9].

Definition 2 (Proposition). The space of almost automorphic distributions, denoted by $\mathscr{B}'_{aa}$, is the space of $T \in \mathscr{B}'$ satisfying one of the following equivalent statements
(1) $T * \varphi \in \mathscr{C}_{aa}$, $\forall\, \varphi \in \mathscr{D}$.
(2) $\exists\, k \in \mathbb{Z}_+$ and $g_j \in \mathscr{C}_{aa}$, $0 \leqslant j \leqslant k$, such that $T = \sum_{j=0}^{k} g_j^{(j)}$.

The space of bounded distributions vanishing at infinity, denoted by $\mathscr{B}'_{+,0}$, is the space of $Q \in \mathscr{B}'$ satisfying

$$\lim_{\omega \to +\infty} \langle \tau_\omega Q, \varphi\rangle := \lim_{\omega \to +\infty} \langle Q, \tau_{-\omega}\varphi\rangle = 0 \quad (\forall\, \varphi \in \mathscr{D}),$$

where $\tau_\omega \varphi(\cdot) := \varphi(\cdot + \omega)$, $\omega \in \mathbb{R}$.

Theorem 1. *The space of asymptotically almost automorphic distributions, denoted by $\mathscr{B}'_{aaa}$, is the space of $T \in \mathscr{B}'$ satisfying one of the following equivalent statements*
(1) $\exists\, P \in \mathscr{B}'_{aa}$, $\exists\, Q \in \mathscr{B}'_{+,0}$ *such that* $T = P + Q$ *on* $\mathbb{J}$.
(2) $T * \varphi \in \mathscr{C}_{aaa}$, $\forall\, \varphi \in \mathscr{D}$.
(3) $\exists\, k \in \mathbb{Z}_+$ *and* $f_j \in \mathscr{C}_{aaa}$, $0 \leqslant j \leqslant k$, *such that* $T = \sum_{j=0}^{k} f_j^{(j)}$.
(4) $\exists\, (\theta_m)_{m \in \mathbb{N}} \subset \mathscr{B}_{aaa}$ *such that* $\lim_{m \to +\infty} \theta_m = T$ *in* $\mathscr{B}'$.

Notation 2. If $T \in \mathscr{B}'_{aaa}$ and $T = P + Q$ on $\mathbb{J}$, since this decomposition is unique by [9, Proposition 12], the distribution P is called the principal term of T, the distribution Q is called the corrective term of T, we denote them respectively T_{aa} and T_{cor}. This is summarized by the notation $T = (T_{aa} + T_{cor}) \in \mathscr{B}'_{aaa}$.

3. Asymptotically Almost Automorphic Generalized Functions

We introduce and study an algebra of asymptotically almost automorphic generalized functions.

Let $I :=]0,1]$, $(u_\varepsilon)_\varepsilon \in (\mathscr{D}_{L^p}(\mathbb{I}))^I$, $p \in [1,+\infty]$, $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$, then the notation

$$|u_\varepsilon|_{k,p,\mathbb{I}} = O(\varepsilon^m),\ \varepsilon \to 0 \iff (\exists\, c > 0)\,(\exists\, \varepsilon_0 \in I)\,(\forall\, \varepsilon < \varepsilon_0) \quad |u_\varepsilon|_{k,p,\mathbb{I}} \leqslant c\varepsilon^m.$$

The space of moderate elements is denoted and defined by

$$\mathscr{M}_{aaa} = \big\{(u_\varepsilon)_\varepsilon \in (\mathscr{B}_{aaa})^I : \forall\, k \in \mathbb{Z}_+,\ \exists\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty,\mathbb{R}} = O(\varepsilon^{-m}),\ \varepsilon \to 0\big\},$$

and the space of null elements by

$$\mathscr{N}_{aaa} = \big\{(u_\varepsilon)_\varepsilon \in (\mathscr{B}_{aaa})^I : \forall\, k \in \mathbb{Z}_+,\ \forall\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty,\mathbb{R}} = O(\varepsilon^{m}),\ \varepsilon \to 0\big\}.$$

Some properties of $\mathscr{M}_{aaa}$ and $\mathscr{N}_{aaa}$ are given in the following results.

Proposition 3. (1) *We have the null characterization of* $\mathscr{N}_{aaa}$, *i. e.*

$$\mathscr{N}_{aaa} = \big\{(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa} : \forall\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{0,\infty,\mathbb{R}} = O(\varepsilon^{m}),\ \varepsilon \to 0\big\}.$$

(2) *The space* $\mathscr{M}_{aaa}$ *is an algebra stable under translation and derivation.*
(3) *The space* $\mathscr{N}_{aaa}$ *is an ideal of* $\mathscr{M}_{aaa}$.

$\lhd$ (1) The proof is based on the following Landau–Kolmogorov inequality

$$\big\|f^{(p)}\big\|_{L^\infty(\mathbb{R})} \leqslant 2\pi \big\|f\big\|_{L^\infty(\mathbb{R})}^{1-\frac{p}{n}} \big\|f^{(n)}\big\|_{L^\infty(\mathbb{R})}^{\frac{p}{n}},$$

where $0 < p < n \in \mathbb{Z}_+$ and f is of class $\mathscr{C}^n(\mathbb{R})$. Let $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$, i. e.

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, m \in \mathbb{Z}_+)\,(\exists\, c > 0)\,(\exists\, \varepsilon_1 \in I)\,(\forall\, \varepsilon < \varepsilon_1) \quad |u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c\varepsilon^{-m}, \tag{12}$$

and it satisfies the estimate of order zero, i. e.

$$(\forall\, m_2 \in \mathbb{Z}_+)\,(\exists\, c_2 > 0)\,(\exists\, \varepsilon_2 \in I)\,(\forall\, \varepsilon < \varepsilon_2) \quad |u_\varepsilon|_{0,\infty,\mathbb{R}} \leqslant c_2\varepsilon^{m_2}. \tag{13}$$

The Landau–Kolmogorov inequality for $p = j$, $n = 2j$, (12) and (13) give $\forall\, k \in \mathbb{Z}_+$,

$$|u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant \sum_{j \leqslant k} 2\pi \big\|u_\varepsilon\big\|_{L^\infty(\mathbb{R})}^{1-\frac{1}{2}} \big\|u_\varepsilon^{(2j)}\big\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}$$

$$\leqslant 2\pi\big(|u_\varepsilon|_{0,\infty,\mathbb{R}}\big)^{\frac{1}{2}} \sum_{l \leqslant 2k} \big\|u_\varepsilon^{(l)}\big\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \leqslant 2\pi c^{\frac{1}{2}} c_2^{\frac{1}{2}} \varepsilon^{\frac{-m}{2}} \varepsilon^{\frac{m_2}{2}}.$$

Taking $m_2 \in \mathbb{Z}_+$ such that $m_0 = 2\big(-\frac{m}{2} + \frac{m_2}{2}\big) \in \mathbb{Z}_+$, then we obtain

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m_0 \in \mathbb{Z}_+)\,(\exists\, c = 2\pi c^{\frac{1}{2}} c_2^{\frac{1}{2}} > 0)\,(\exists\, \varepsilon_0 = \inf(\varepsilon_1, \varepsilon_2) \in I)\,(\forall\, \varepsilon < \varepsilon_0)\ |u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c\varepsilon^{m_0}.$$

Which gives $(u_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$.

(2) The stability under translation and derivation of the space $\mathscr{M}_{aaa}$ is obvious. Let $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$, i. e. they satisfy (12) and for $j \in \mathbb{Z}_+$,

$$\left\|(u_\varepsilon v_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leqslant \sum_{i\leqslant j} \frac{j!}{i!(j-i)!} \left\|u_\varepsilon^{(i)}\right\|_{L^\infty(\mathbb{R})} \left\|v_\varepsilon^{(j-i)}\right\|_{L^\infty\left(\mathbb{R}\right)}$$
$$\leqslant |u_\varepsilon|_{j,\infty,\mathbb{R}}\, |v_\varepsilon|_{j,\infty,\mathbb{R}} \sum_{i\leqslant j} \frac{j!}{i!(j-i)!} \leqslant 2^j c_1 c_2 \varepsilon^{-m_1} \varepsilon^{-m_2},$$

consequently,

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, m = (m_1 + m_2) \in \mathbb{Z}_+) \quad |u_\varepsilon v_\varepsilon|_{k,\infty,\mathbb{R}} = O(\varepsilon^{-m}),\ \varepsilon \to 0.$$

Which shows that $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$.

(3) Let $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(u_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, i. e. $(v_\varepsilon)_\varepsilon$ satisfies (12) and $(u_\varepsilon)_\varepsilon$ satisfies

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m' \in \mathbb{Z}_+)\,(\exists\, c' > 0)\,(\exists\, \varepsilon_1' \in I)\,(\forall\, \varepsilon < \varepsilon_1') \quad |u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c' \varepsilon^{m'}. \tag{14}$$

Since the family of the norms $|\cdot|_{k,\infty,\mathbb{R}}$ is compatible with the algebraic structure of $\mathscr{B}$, i. e. $\forall\, k \in \mathbb{Z}_+$, $\exists\, c_k > 0$ such that

$$|u_\varepsilon v_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c_k |u_\varepsilon|_{k,\infty,\mathbb{R}}\, |v_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c_k c c' \varepsilon^{-m} \varepsilon^{m'}.$$

Take $m' \in \mathbb{Z}_+$ such that $m_0 = (-m + m') \in \mathbb{Z}_+$, so we obtain

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m_0 \in \mathbb{Z}_+)\,(\exists\, C = c_k c c' > 0)\,(\exists\, \varepsilon_0 = \inf\left(\varepsilon_1, \varepsilon_1'\right) \in I)\,(\forall\, \varepsilon < \varepsilon_0)\ \left|u_\varepsilon v_\varepsilon\right|_{k,\infty,\mathbb{R}} \leqslant C\varepsilon^{m_0},$$

which implies $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$. $\rhd$

Definition 3. The algebra of asymptotically almost automorphic generalized functions is denoted and defined as the quotient algebra

$$\mathscr{G}_{aaa} := \frac{\mathscr{M}_{aaa}}{\mathscr{N}_{aaa}}$$

Example 1. We have $\mathscr{G}_{aap} \subsetneq \mathscr{G}_{aaa}$, where $\mathscr{G}_{aap}$ is the algebra of asymptotically almost periodic generalized functions of [20]. Let $\rho \in \mathscr{S}$ such that

$$\int\limits_{\mathbb{R}} \rho(x)\, dx = 1 \quad \text{and} \quad \int\limits_{\mathbb{R}} x^k \rho(x)\, dx = 0 \quad (\forall\, k \in \mathbb{N}).$$

Set $\rho_\varepsilon(\cdot) := \frac{1}{\varepsilon}\, \rho(\frac{\cdot}{\varepsilon})$, $\varepsilon > 0$.

Define the following maps

$$\begin{array}{rccc} \iota_{aaa}: & \mathscr{B}'_{aaa} & \longrightarrow & \mathscr{G}_{aaa} \\ & T & \longmapsto & (T * \rho_\varepsilon)_\varepsilon + \mathscr{N}_{aaa} \end{array}$$

$$\begin{array}{rccc} \sigma_{aaa}: & \mathscr{B}_{aaa} & \longrightarrow & \mathscr{G}_{aaa} \\ & f & \longmapsto & (f)_\varepsilon + \mathscr{N}_{aaa} \end{array}$$

$$\begin{array}{rccc} \gamma_{aaa}: & \mathscr{B}_{aaa} & \longrightarrow & \mathscr{B}'_{aaa} \\ & f & \longmapsto & f \end{array}$$

Proposition 4. *The following diagram of linear embeddings*

$$\begin{array}{ccc} \mathscr{B}_{aaa} & \overset{\gamma_{aaa}}{\longrightarrow} & \mathscr{B}'_{aaa} \\ & \sigma_{aaa} \searrow & \downarrow \iota_{aaa} \\ & & \mathscr{G}_{aaa} \end{array}$$

is commutative.

$\lhd$ For $f \in \mathscr{B}_{aaa}$, we have $\gamma_{aaa}(f) \in \mathscr{B}_{aaa} \subset \mathscr{C}_{aaa}$. We conclude from ([9, Example 3-(1)]) that $\gamma_{aaa}(f) \in \mathscr{B}'_{aaa}$. Let $T \in \mathscr{B}'_{aaa}$, by the characterization of an asymptotically almost automorphic distribution $\exists\, (f_i)_{i\leqslant m} \subset \mathscr{C}_{aaa}$ such that $T = \sum_{i\leqslant m} f_i^{(i)}$. If $j \in \mathbb{Z}_+$,

$$\left|(T * \rho_\varepsilon)^{(j)}(x)\right| \leqslant \sum_{i\leqslant m} \frac{1}{\varepsilon^{i+j}} \int\limits_{\mathbb{R}} \left|f_i(x-\varepsilon y)\rho^{(i+j)}(y)\right| dy \leqslant \sum_{i\leqslant m} \frac{1}{\varepsilon^{i+j}} \left\|f_i\right\|_{L^\infty(\mathbb{R})} \int\limits_{\mathbb{R}} \left|\rho^{(i+j)}(y)\right| dy,$$

consequently, there exists $c_{m,j} > 0$ such that

$$\left\|(T * \rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leqslant \frac{c_{m,j}}{\varepsilon^{m+j}},$$

hence,

$$(\forall\, k \in \mathbb{Z}_+)\, (\exists\, m_1 \in \mathbb{Z}_+) \quad |T * \rho_\varepsilon|_{k,\infty,\mathbb{R}} = O\left(\varepsilon^{-m_1}\right), \quad \varepsilon \to 0,$$

which gives that $(T * \rho_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$. The linearity of ι_{aaa} results from the fact that the convolution is linear. If $(T * \rho_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, then

$$(\forall\, m' \in \mathbb{Z}_+)\, (\exists\, c' > 0)\, (\exists\, \varepsilon_1' \in I)\, (\forall\, \varepsilon < \varepsilon_1') \quad \|T * \rho_\varepsilon\|_{L^\infty(\mathbb{R})} \leqslant c'\varepsilon^{m'}. \tag{15}$$

Due to regularization we have

$$\langle T, \psi\rangle = \lim_{\varepsilon\to 0} \int (T * \rho_\varepsilon)(x)\psi(x)\, dx, \quad \psi \in \mathscr{D}_{L^1}(\mathbb{R}).$$

From (15), it holds $\forall\, m' \in \mathbb{Z}_+$, $\exists\, c' > 0$, $\exists\, \varepsilon_1' \in I$, $\forall\, \varepsilon < \varepsilon_1'$,

$$\left|\int (T * \rho_\varepsilon)(x)\psi(x)\, dx\right| \leqslant \left\|\psi\right\|_{L^1(\mathbb{R})} c'\varepsilon^{m'},$$

consequently, when $\varepsilon \to 0$ we obtain $\langle T, \psi\rangle = 0$, $\forall\, \psi \in \mathscr{D}_{L^1}(\mathbb{R})$, therefore ι_{aaa} is injective. Finally,

$$\iota_{aaa}\big(T^{(j)}\big) = \big(T^{(j)} * \rho_\varepsilon\big)_\varepsilon + \mathscr{N}_{aaa} = \big(T * \rho_\varepsilon\big)_\varepsilon^{(j)} + \mathscr{N}_{aaa} = \big(\iota_{aaa}(T)\big)^{(j)} \quad (\forall\, j \in \mathbb{Z}_+),$$

i. e. the embedding ι_{aaa} commutes with derivatives.

Let $f \in \mathscr{B}_{aaa}$, we have to show that $(f * \rho_\varepsilon - f)_\varepsilon \in \mathscr{N}_{aaa}$. By Taylor's formula, for $\theta \in]0, 1[$ and $m \in \mathbb{N}$, we set for $j \in \mathbb{Z}_+$, that $(f^{(j)} * \rho_\varepsilon - f^{(j)})(x)$ equals

$$\int\limits_{\mathbb{R}} \sum_{l=1}^{m-1} \frac{(-\varepsilon y)^l}{l!} f^{(l+j)}(x)\rho(y)\, dy + \int\limits_{\mathbb{R}} \frac{(-\varepsilon y)^m}{m!} f^{(m+j)}(x - \theta\varepsilon y)\rho(y)\, dy,$$

and then

$$\begin{aligned} \left\|f^{(j)} * \rho_\varepsilon - f^{(j)}\right\|_{L^\infty(\mathbb{R})} &\leqslant \frac{\varepsilon^m}{m!} \sup_{x\in\mathbb{R}} \int\limits_{\mathbb{R}} \left|(-y)^m\right| \left|f^{(m+j)}(x - \theta\varepsilon y)\right| \left|\rho(y)\right| dy, \\ &\leqslant \frac{\varepsilon^m}{m!} \left\|f^{(m+j)}\right\|_{L^\infty(\mathbb{R})} \left\|y^m\rho\right\|_{L^1(\mathbb{R})}. \end{aligned}$$

Hence, $(\forall\, k \in \mathbb{Z}_+)$ $(\forall\, m \in \mathbb{N})$

$$\left|f * \rho_\varepsilon - f\right|_{k,\infty,\mathbb{R}} \leqslant \frac{\varepsilon^m}{m!}\,|f|_{m+k,\infty,\mathbb{R}}\,\left\|y^m \rho\right\|_{L^1(\mathbb{R})},$$

which means that $(f * \rho_\varepsilon - f)_\varepsilon \in \mathscr{N}_{aaa}$. $\rhd$

Remark 2. The application σ_{aaa} is not only a linear embedding but it is also an algebraic embedding.

Recall the algebra of almost automorphic generalized functions, see [22], denoted and defined by

$$\mathscr{G}_{aa} := \frac{\mathscr{M}_{aa}}{\mathscr{N}_{aa}},$$

where

$$\mathscr{M}_{aa} := \left\{(u_\varepsilon)_\varepsilon \in (\mathscr{B}_{aa})^I : \forall\, k \in \mathbb{Z}_+,\ \exists\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty,\mathbb{R}} = O(\varepsilon^{-m}),\ \varepsilon \to 0\right\}.$$

$$\mathscr{N}_{aa} := \left\{(u_\varepsilon)_\varepsilon \in (\mathscr{B}_{aa})^I : \forall\, k \in \mathbb{Z}_+,\ \forall\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty,\mathbb{R}} = O(\varepsilon^{m}),\ \varepsilon \to 0\right\}.$$

The algebra of L^p-generalized functions on $\mathbb{I}$, $1 \leqslant p \leqslant +\infty$, see [23], is defined as the quotient algebra

$$\mathscr{G}_{L^p}(\mathbb{I}) := \frac{\mathscr{M}_{L^p}(\mathbb{I})}{\mathscr{N}_{L^p}(\mathbb{I})},$$

where

$$\mathscr{M}_{L^p}(\mathbb{I}) := \left\{(u_\varepsilon)_\varepsilon \in (\mathscr{D}_{L^p}(\mathbb{I}))^I : \forall\, k \in \mathbb{Z}_+,\ \exists\, m > 0,\ |u_\varepsilon|_{k,p,\mathbb{I}} = O(\varepsilon^{-m}),\ \varepsilon \to 0\right\}.$$

$$\mathscr{N}_{L^p}(\mathbb{I}) := \left\{(u_\varepsilon)_\varepsilon \in (\mathscr{D}_{L^p}(\mathbb{I}))^I : \forall\, k \in \mathbb{Z}_+,\ \forall\, m > 0,\ |u_\varepsilon|_{k,p,\mathbb{I}} = O(\varepsilon^{m}),\ \varepsilon \to 0\right\}.$$

Notation 3. Denote $\mathscr{G}_{L^\infty}(\mathbb{I}) := \mathscr{G}_{\mathscr{B}}(\mathbb{I})$, $\mathscr{G}_{\mathscr{B}} := \mathscr{G}_{\mathscr{B}}(\mathbb{R})$ and $\mathscr{G}_{L^1} := \mathscr{G}_{L^1}(\mathbb{R})$.

For $\omega \in \mathbb{R}$, the translate $\tau_\omega \widetilde{u}$ of $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}$ is defined by

$$\tau_\omega \widetilde{u} := \left[(\tau_\omega u_\varepsilon)_\varepsilon\right].$$

For $j \in \mathbb{Z}_+$, the derivation $\widetilde{u}^{(j)}$ of $\widetilde{u}$ is defined by

$$\widetilde{u}^{(j)} := \left[\left(u_\varepsilon^{(j)}\right)_\varepsilon\right].$$

Let $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}$, the product $\widetilde{u} \times \widetilde{v}$ is defined by

$$\widetilde{u} \times \widetilde{v} := \left[(u_\varepsilon v_\varepsilon)_\varepsilon\right].$$

Let $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathscr{G}_{L^1}$, the convolution $\widetilde{u} * \widetilde{v}$ is defined by

$$\widetilde{u} * \widetilde{v} := \left[(u_\varepsilon * v_\varepsilon)_\varepsilon\right].$$

The following results lift the results of Proposition 2 to $\mathscr{G}_{aaa}$.

Proposition 5. (1) *$\mathscr{G}_{aaa}$ is a subalgebra of $\mathscr{G}_{\mathscr{B}}$ stable under translation and derivation.*
(2) $\mathscr{G}_{aaa} \times \mathscr{G}_{aa} \subset \mathscr{G}_{aaa}$.
(3) $\mathscr{G}_{aaa} * \mathscr{G}_{L^1} \subset \mathscr{G}_{aaa}$.

$\lhd$ (1) From Proposition 3 (2), we deduce that $\mathscr{G}_{aaa}$ is an algebra stable under translation and derivation. Let $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$, satisfies (12) and as $\mathscr{B}_{aaa} \subset \mathscr{B}$, hence $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{\mathscr{B}}$. If $(u_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, in the same way we prove that $(u_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{B}}$.

(2) Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{aaa}$ and $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathscr{G}_{aa}$. As $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{aa}$, so $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ satisfy the estimate (12). In view of Proposition 2 (2) it follows that for all $\varepsilon \in I$, $u_\varepsilon v_\varepsilon \in \mathscr{B}_{aaa}$, and for every $j \in \mathbb{Z}_+$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leqslant 2^j \left| u_\varepsilon \right|_{j,\infty,\mathbb{R}} \left| v_\varepsilon \right|_{j,\infty,\mathbb{R}} \leqslant 2^j c_1 c_2 \varepsilon^{-m_1} \varepsilon^{-m_2},$$

therefore, $(\forall k \in \mathbb{Z}_+)$ $(\exists m = (m_1 + m_2) \in \mathbb{Z}_+)$ $|u_\varepsilon v_\varepsilon|_{k,\infty,\mathbb{R}} = O(\varepsilon^{-m})$, $\varepsilon \to 0$, i. e., $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$. The product $\widetilde{u} \times \widetilde{v}$ is independent on the representatives. Indeed, suppose that $(u'_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(v'_\varepsilon)_\varepsilon \in \mathscr{M}_{aa}$ are others representatives of $\widetilde{u}$ and $\widetilde{v}$ respectively, for $j \in \mathbb{Z}_+$,

$$\begin{aligned}\left\| \left(u_\varepsilon v_\varepsilon - u'_\varepsilon v'_\varepsilon\right)^{(j)} \right\|_{L^\infty(\mathbb{R})} &= \left\| \left(u_\varepsilon v_\varepsilon - u'_\varepsilon v_\varepsilon + u'_\varepsilon v_\varepsilon - u'_\varepsilon v'_\varepsilon\right)^{(j)} \right\|_{L^\infty(\mathbb{R})} \\ &\leqslant \left\| \left((u_\varepsilon - u'_\varepsilon) v_\varepsilon\right)^{(j)} \right\|_{L^\infty(\mathbb{R})} + \left\| \left(u'_\varepsilon (v_\varepsilon - v'_\varepsilon)\right)^{(j)} \right\|_{L^\infty(\mathbb{R})} \\ &\leqslant 2^j \Big(\left| u_\varepsilon - u'_\varepsilon \right|_{j,\infty,\mathbb{R}} \left| v_\varepsilon \right|_{j,\infty,\mathbb{R}} + \left| u'_\varepsilon \right|_{j,\infty,\mathbb{R}} \left| v_\varepsilon - v'_\varepsilon \right|_{j,\infty,\mathbb{R}} \Big),\end{aligned}$$

since $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, $(v_\varepsilon - v'_\varepsilon)_\varepsilon \in \mathscr{N}_{aa}$, then

$$(\forall k \in \mathbb{Z}_+)\, (\forall m \in \mathbb{Z}_+) \quad \left| u_\varepsilon v_\varepsilon - u'_\varepsilon v'_\varepsilon \right|_{k,\infty,\mathbb{R}} = O(\varepsilon^m),$$

as $\varepsilon \to 0$, which implies that $(u_\varepsilon v_\varepsilon - u'_\varepsilon v'_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$.

(3) Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{aaa}$ and $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathscr{G}_{L^1}$, so $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{L^1}$, that is $(u_\varepsilon)_\varepsilon$ satisfies the estimate 12, and $(v_\varepsilon)_\varepsilon$ satisfies

$$(\forall k \in \mathbb{Z}_+)\, (\exists m_1 > 0)\ (\exists c > 0)\, (\exists \varepsilon_0 \in I)\, (\forall \varepsilon < \varepsilon_0) \quad |v_\varepsilon|_{k,1,\mathbb{R}} \leqslant c\varepsilon^{-m_1}.$$

Proposition 2 (3) gives for all $\varepsilon \in I$, $u_\varepsilon * v_\varepsilon \in \mathscr{B}_{aaa}$. Due to Young inequality we obtain

$$\left\| (u_\varepsilon * v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leqslant \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \left\| v_\varepsilon \right\|_{L^1(\mathbb{R})}.$$

For every $k \in \mathbb{Z}_+$,

$$\left| u_\varepsilon * v_\varepsilon \right|_{k,\infty,\mathbb{R}} \leqslant \sum_{j \leqslant k} \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \left\| v_\varepsilon \right\|_{L^1(\mathbb{R})} \leqslant \left| u_\varepsilon \right|_{k,\infty,\mathbb{R}} \left| v_\varepsilon \right|_{0,1,\mathbb{R}},$$

consequently,

$$(\forall k \in \mathbb{Z}_+)\, (\exists m \in \mathbb{Z}_+) \quad \left| u_\varepsilon * v_\varepsilon \right|_{k,\infty,\mathbb{R}} = O(\varepsilon^{-m}),\ \varepsilon \to 0,$$

so $(u_\varepsilon * v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$. The convolution $\widetilde{u} * \widetilde{v}$ does not depend on the representatives. Indeed, suppose that $(u'_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(v'_\varepsilon)_\varepsilon \in \mathscr{M}_{L^1}$ are others representatives of $\widetilde{u}$ and $\widetilde{v}$ respectively, for every $k \in \mathbb{Z}_+$,

$$\begin{aligned}\left| u_\varepsilon * v_\varepsilon - u'_\varepsilon * v'_\varepsilon \right|_{k,\infty,\mathbb{R}} &= \left| u_\varepsilon * v_\varepsilon - u'_\varepsilon * v_\varepsilon + u'_\varepsilon * v_\varepsilon - u'_\varepsilon * v'_\varepsilon \right|_{k,\infty,\mathbb{R}} \\ &\leqslant \left| u_\varepsilon - u'_\varepsilon \right|_{k,\infty,\mathbb{R}} \left| v_\varepsilon \right|_{0,1,\mathbb{R}} + \left| u'_\varepsilon \right|_{k,\infty,\mathbb{R}} \left| v_\varepsilon - v'_\varepsilon \right|_{0,1,\mathbb{R}},\end{aligned}$$

as $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, $(v_\varepsilon - v'_\varepsilon)_\varepsilon \in \mathscr{N}_{L^1}$, then

$$(\forall k \in \mathbb{Z}_+)\, (\forall m \in \mathbb{Z}_+) \quad \left| u_\varepsilon * v_\varepsilon - u'_\varepsilon * v'_\varepsilon \right|_{k,\infty,\mathbb{R}} = O(\varepsilon^m),$$

as $\varepsilon \to 0$, which means that $(u_\varepsilon * v_\varepsilon - u'_\varepsilon * v'_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$. $\rhd$

4. A Generalized Seeley Theorem

We give a result on the extension operators in the context of generalized functions. It is needed in the proof of the decomposition of an asymptotically almost automorphic generalized function. Let's first recall a technical Lemma, see [24].

Lemma 1. *There are two sequences of real numbers $(a_l)_{l\in\mathbb{Z}_+}$ and $(b_l)_{l\in\mathbb{Z}_+}$ such that*

(1) $b_l < 0$, $\forall\, l \in \mathbb{Z}_+$.

(2) $\sum_{l=0}^{+\infty} |a_l||b_l|^n < +\infty$, $\forall\, n \in \mathbb{Z}_+$.

(3) $\sum_{l=0}^{+\infty} a_l b_l^n = 1$, $\forall\, n \in \mathbb{Z}_+$.

(4) $b_l \to -\infty$, $l \to +\infty$.

Define the space

$$\mathscr{B}_{+,0}(\mathbb{I}) := \left\{\varphi \in \mathscr{B}(\mathbb{I}) : \forall\, j \in \mathbb{Z}_+,\ \lim_{x\to+\infty} \varphi^{(j)}(x) = 0\right\}.$$

The algebra of bounded generalized functions vanishing at infinity on $\mathbb{I}$ is defined by

$$\mathscr{G}_{+,0}(\mathbb{I}) := \frac{\mathscr{M}_{+,0}(\mathbb{I})}{\mathscr{N}_{+,0}(\mathbb{I})},$$

where

$$\mathscr{M}_{+,0}(\mathbb{I}) := \left\{(u_\varepsilon)_\varepsilon \in (\mathscr{B}_{+,0}(\mathbb{I}))^I : \forall\, k \in \mathbb{Z}_+,\ \exists\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty,\mathbb{I}} = O(\varepsilon^{-m}),\ \varepsilon \to 0\right\}.$$

$$\mathscr{N}_{+,0}(\mathbb{I}) := \left\{(u_\varepsilon)_\varepsilon \in (\mathscr{B}_{+,0}(\mathbb{I}))^I : \forall\, k \in \mathbb{Z}_+,\ \forall\, m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty,\mathbb{I}} = O(\varepsilon^{m}),\ \varepsilon \to 0\right\}.$$

Theorem 2. *The linear extension operator $\widetilde{E} : \mathscr{G}_{\mathscr{B}}(\mathbb{J}) \to \mathscr{G}_{\mathscr{B}}(\mathbb{R})$, $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \mapsto \widetilde{E}\widetilde{u} = [(Eu_\varepsilon)_\varepsilon]$, where*

$$Eu_\varepsilon(x) := \begin{cases} u_\varepsilon(x), & \text{if } x \geqslant 0,\\ \sum\limits_{l=0}^{+\infty} a_l u_\varepsilon(b_l x), & \text{if } x < 0,\end{cases}$$

is well defined and we have $\widetilde{E}\tilde{u}_{|\mathbb{J}} = \tilde{u}$. In particular, $\forall\, \tilde{u} \in \mathscr{G}_{+,0}(\mathbb{J})$, $\widetilde{E}\tilde{u} \in \mathscr{G}_{+,0}(\mathbb{R})$.

$\lhd$ Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}(\mathbb{J})$, and $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{\mathscr{B}}(\mathbb{J})$ be a representative of $\widetilde{u}$. So $\forall\, \varepsilon \in I$, $Eu_\varepsilon \in \mathscr{B}(\mathbb{R})$ and $Eu_{\varepsilon|\mathbb{J}} = u_\varepsilon$. Indeed, if $x < 0$, then $b_l x > 0$, $\forall\, l \in \mathbb{Z}_+$ in view of Lemma 1 (1). Moreover according to Lemma 1 (2) and as $u_\varepsilon \in \mathscr{B}(\mathbb{J})$, $\forall\, \varepsilon \in I$, hence $\forall\, n \in \mathbb{Z}_+$, $\forall\, \varepsilon \in I$, $\forall\, x < 0$,

$$\left|(Eu_\varepsilon)^{(n)}(x)\right| \leqslant \left\|u_\varepsilon^{(n)}\right\|_{L^\infty(\mathbb{J})} \sum_{l=0}^{+\infty} |a_l|\,|b_l|^n < +\infty, \tag{16}$$

consequently, $\forall\, n \in \mathbb{Z}_+$, the series

$$(Eu_\varepsilon)^{(n)}(x) = \sum_{l=0}^{+\infty} a_l b_l^n u_\varepsilon^{(n)}(b_l x), \quad \varepsilon \in I,$$

absolutely converge. Furthermore due to Lemma 1 (3),

$$\lim_{\substack{x\to 0\\ <}} (Eu_\varepsilon)^{(n)}(x) = \sum_{l=0}^{+\infty} a_l b_l^n \lim_{\substack{x\to 0\\ <}} u_\varepsilon^{(n)}(b_l x) = u_\varepsilon^{(n)}(0) \sum_{l=0}^{+\infty} a_l b_l^n = u_\varepsilon^{(n)}(0),$$

so $\forall\,\varepsilon \in I$, $Eu_\varepsilon \in \mathscr{E}(\mathbb{R})$. As $\forall\,\varepsilon \in I$, $u_\varepsilon \in \mathscr{B}(\mathbb{J})$ and by (16), it follows that $\forall\,n \in \mathbb{Z}_+$, $\forall\,\varepsilon \in I$, $(Eu_\varepsilon)^{(n)} \in L^\infty(\mathbb{R})$. In order to show that $(Eu_\varepsilon)_\varepsilon \in \mathscr{M}_{\mathscr{B}}(\mathbb{R})$, we prove the following estimate

$$(\forall\,\varepsilon \in I)\,(\forall\,k \in \mathbb{Z}_+)\,(\exists\,C_k > 0) \quad |Eu_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant C_k |u_\varepsilon|_{k,\infty,\mathbb{J}}. \tag{17}$$

Indeed, it is clear that

$$\big\|(Eu_\varepsilon)^{(n)}\big\|_{L^\infty(\mathbb{J})} = \big\|u_\varepsilon^{(n)}\big\|_{L^\infty(\mathbb{J})}\,,$$

and the estimate (16), gives

$$\|(Eu_\varepsilon)^{(n)}\big\|_{L^\infty(\mathbb{R}\setminus\mathbb{J})} \leqslant \big\|u_\varepsilon^{(n)}\big\|_{L^\infty(\mathbb{J})} \sum_{l=0}^{+\infty} |a_l|\,|b_l|^n,$$

therefore

$$\big\|(Eu_\varepsilon)^{(n)}\big\|_{L^\infty(\mathbb{R})} \leqslant C_n \big\|u_\varepsilon^{(n)}\big\|_{L^\infty(\mathbb{J})}\,,$$

where $C_n = \max\big(1, \sum_{l=0}^{+\infty} |a_l||b_l|^n\big)$. So

$$(\forall\,k \in \mathbb{Z}_+)\,\Big(\exists\,C_k := \sum_{n\leqslant k} C_n < +\infty\Big)\,(\forall\,\varepsilon \in I)\,(u_\varepsilon \in \mathscr{B}(\mathbb{J}))$$

$$\big|Eu_\varepsilon\big|_{k,\infty,\mathbb{R}} \leqslant C_k \big|u_\varepsilon\big|_{k,\infty,\mathbb{J}}\,, \tag{18}$$

this implies $(Eu_\varepsilon)_\varepsilon \in \mathscr{M}_{\mathscr{B}}(\mathbb{R})$. The definition of $\tilde{E}\tilde{u}$ is independent on representatives. Indeed, if $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ are representatives of $\tilde{u}$, hence by (18),

$$(\forall\,\varepsilon \in I)\,(\forall\,k \in \mathbb{Z}_+)\,(\exists\,C_k > 0) \quad \big|Eu_\varepsilon - Ev_\varepsilon\big|_{k,\infty,\mathbb{R}} \leqslant C_k \big|u_\varepsilon - v_\varepsilon\big|_{k,\infty,\mathbb{J}}\,.$$

As $(u_\varepsilon - v_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{J})$ then

$$(\forall\,k \in \mathbb{Z}_+)\,(\forall\,m > 0) \quad \big|Eu_\varepsilon - Ev_\varepsilon\big|_{k,\infty,\mathbb{R}} = O(\varepsilon^m),\ \varepsilon \to 0,$$

which shows that $(Eu_\varepsilon - Ev_\varepsilon)_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{R})$.

We have $\widetilde{E}\widetilde{u}_{|\mathbb{J}} = \widetilde{u}$ in $\mathscr{G}_{\mathscr{B}}(\mathbb{J})$. Indeed, as $\widetilde{E}\tilde{u} = [(Eu_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}(\mathbb{R})$ and $\tilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}(\mathbb{J})$, therefore

$$\widetilde{E}\tilde{u}_{|\mathbb{J}} - \tilde{u} := \big[(Eu_{\varepsilon|\mathbb{J}})_\varepsilon\big] - \tilde{u} = \big[(u_\varepsilon)_\varepsilon\big] - \tilde{u} = \tilde{u} - \tilde{u} = 0 \quad \text{in} \quad \mathscr{G}_{\mathscr{B}}(\mathbb{J}).$$

If $\tilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{+,0}(\mathbb{J}) \subset \mathscr{G}_{\mathscr{B}}(\mathbb{J})$, then $\widetilde{E}\tilde{u} = [(Eu_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}(\mathbb{R})$. So $\forall\,\varepsilon \in I$, $Eu_\varepsilon \in \mathscr{B}(\mathbb{R})$. The fact that $\forall\,\varepsilon \in I$, $Eu_\varepsilon = u_\varepsilon$ on $\mathbb{J}$, and $\forall\,\varepsilon \in I$, $u_\varepsilon \in \mathscr{B}_{+,0}(\mathbb{J})$ implies $\lim_{x\to+\infty} Eu_\varepsilon(x) = 0$, i. e. $\forall\,\varepsilon \in I$, $Eu_\varepsilon \in \mathscr{C}_{+,0}$. By [9, Proposition 5 (5)], we obtain that

$$(\forall\,\varepsilon \in I) \quad Eu_\varepsilon \in \mathscr{C}_{+,0} \cap \mathscr{B}(\mathbb{R}) = \mathscr{B}_{+,0}(\mathbb{R}),$$

it follows that $\widetilde{E}\tilde{u} \in \mathscr{G}_{+,0}(\mathbb{R})$. $\rhd$

5. The Decomposition

In this section we show that an asymptotically almost automorphic generalized function is uniquely decomposed as in the classical case.

Theorem 3. *Let $\widetilde{u} \in \mathscr{G}_{aaa}(\mathbb{R})$ then there exist $\widetilde{v} \in \mathscr{G}_{aa}(\mathbb{R})$ and $\widetilde{w} \in \mathscr{G}_{+,0}(\mathbb{R})$ such that $\widetilde{u} = \widetilde{v} + \widetilde{w}$ on $\mathbb{J}$, and the decomposition is unique on $\mathbb{J}$.*

$\lhd$ Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{aaa}$, so $\forall\, \varepsilon \in I$, $\forall\, j \in \mathbb{Z}_+$, $u_\varepsilon^{(j)} \in \mathscr{C}_{aaa}$. Then there exist $v_{\varepsilon,j} \in \mathscr{C}_{aa}$, $w_{\varepsilon,j} \in \mathscr{C}_{+,0}$, such that $\forall\, j \in \mathbb{N}$, $u_\varepsilon^{(j)} = (v_{\varepsilon,j} + w_{\varepsilon,j}) \in \mathscr{C}_{aaa}$ on $\mathbb{J}$, and for $j = 0$, $u_\varepsilon = v_\varepsilon + w_\varepsilon$ on $\mathbb{J}$. By [9, Proposition 8], it holds that $\forall\, j \in \mathbb{N}$, $v_{\varepsilon,j} = (v_\varepsilon)^{(j)}$ on $\mathbb{R}$ and $w_{\varepsilon,j} = (w_\varepsilon)^{(j)}$ on $\mathbb{J}$, which gives $v_\varepsilon \in \mathscr{B}_{aa}$ and $w_\varepsilon \in \mathscr{B}_{+,0}(\mathbb{J})$. Let's show that $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{aa}$. As $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$, therefore

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, m \in \mathbb{Z}_+)\,(\exists\, c > 0)\,(\exists\, \varepsilon_0 \in I)\,(\forall\, \varepsilon < \varepsilon_0) \quad |u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c\varepsilon^{-m}. \tag{19}$$

Due to [9, Proposition 3 (5)], we obtain

$$(\forall\, j \in \mathbb{Z}_+) \quad \left\| v_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \leqslant \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})}, \tag{20}$$

it follows that

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, m \in \mathbb{Z}_+)\,(\exists\, c > 0)\,(\exists\, \varepsilon_0 \in I)\,(\forall\, \varepsilon < \varepsilon_0) \quad |v_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c\varepsilon^{-m}, \tag{21}$$

this means that $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{aa}$. If $(u_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$ then

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m \in \mathbb{Z}_+)\,(\exists\, c > 0)\,(\exists\, \varepsilon_0 \in I)\,(\forall\, \varepsilon < \varepsilon_0) \quad |u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c\varepsilon^{m}, \tag{22}$$

and from (20) it holds $(v_\varepsilon)_\varepsilon \in \mathscr{N}_{aa}$. Consequently, $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathscr{G}_{aa}$. On the other hand, we have

$$(\forall\, j \in \mathbb{Z}_+) \quad \left\| w_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})} \leqslant \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})} + \left\| v_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})}. \tag{23}$$

The estimates (19), (21) and (23) give

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, m \in \mathbb{Z}_+)\,(\exists\, c > 0)\,(\exists\, \varepsilon_0 \in I)\,(\forall\, \varepsilon < \varepsilon_0) \quad |w_\varepsilon|_{k,\infty,\mathbb{J}} \leqslant 2c\varepsilon^{-m},$$

hence $(w_\varepsilon)_\varepsilon \in \mathscr{M}_{+,0}(\mathbb{J})$. If $(u_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, then $(w_\varepsilon)_\varepsilon \in \mathscr{N}_{+,0}(\mathbb{J})$ follows from (22) and (23). Thus $\widetilde{w} = [(w_\varepsilon)_\varepsilon] \in \mathscr{G}_{+,0}(\mathbb{J})$. By Theorem 2 extending $\widetilde{w} \in \mathscr{G}_{+,0}(\mathbb{J})$ to $\widetilde{E}\widetilde{w} \in \mathscr{G}_{+,0}(\mathbb{R})$ with $\widetilde{E}\widetilde{w} = \widetilde{w}$ on $\mathbb{J}$. Finally, $\widetilde{u} = \widetilde{v} + \widetilde{w}$ on $\mathbb{J}$.

If $\widetilde{u} \in \mathscr{G}_{aaa}$ has two decompositions, i.e.

$$\widetilde{u} = \widetilde{v}_i + \widetilde{w}_i \quad \text{on} \quad \mathbb{J}, \quad i = 1, 2,$$

where $\widetilde{v}_i \in \mathscr{G}_{aa}$ and $\widetilde{w}_i \in \mathscr{G}_{+,0} := \mathscr{G}_{+,0}(\mathbb{R})$. Let $(v_{\varepsilon,i})_\varepsilon \in \mathscr{M}_{aa}$ and $(w_{\varepsilon,i})_\varepsilon \in \mathscr{M}_{+,0}$ be respectively representatives of $\widetilde{v}_i$ and $\widetilde{w}_i$, $i = 1, 2$. So $(v_{\varepsilon,1} - v_{\varepsilon,2})_\varepsilon + (w_{\varepsilon,1} - w_{\varepsilon,2})_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{J})$, i. e.

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m > 0)\,(\exists\, c > 0)\,(\exists\, \varepsilon_0 \in I)\,(\forall\, \varepsilon < \varepsilon_0)$$

$$\left| v_{\varepsilon,1} - v_{\varepsilon,2} + w_{\varepsilon,1} - w_{\varepsilon,2} \right|_{k,\infty,\mathbb{J}} \leqslant c\varepsilon^{m}. \tag{24}$$

Due to [9, Proposition 9], as $\forall\, \varepsilon \in I$, $v_{\varepsilon,i} \in \mathscr{B}_{aa}$, $i = 1, 2$, for any real sequence $(s_m)_{m\in\mathbb{N}}$, such that $s_m \to +\infty$ there exist $(s_{m_{l(\varepsilon)}})_l$ a subsequence of $(s_m)_{m\in\mathbb{N}}$ and $g_{\varepsilon,i}$, $i = 1, 2$, such that $\forall\, x \in \mathbb{R}$, $\forall\, j \in \mathbb{Z}_+$,

$$g_{\varepsilon,i}^{(j)}(x) := \lim_{l\to+\infty} v_{\varepsilon,i}^{(j)}\big(x + s_{m_{l(\varepsilon)}}\big) \quad \text{and} \quad \lim_{l\to+\infty} g_{\varepsilon,i}^{(j)}\big(x - s_{m_{l(\varepsilon)}}\big) = v_{\varepsilon,i}^{(j)}(x).$$

Furthermore, as $\forall\,\varepsilon \in I$, $w_{\varepsilon,i} \in \mathscr{B}_{+,0}$, $i = 1, 2$,

$$\lim_{l\to+\infty} w_{\varepsilon,i}^{(j)}\big(x + s_{m_{l(\varepsilon)}}\big) = 0 \quad (\forall\,x \in \mathbb{R},\ \forall\,j \in \mathbb{Z}_+).$$

By using (24) we have

$$(\forall\,m > 0)\,(\exists\,c > 0)\,(\exists\,\varepsilon_0 \in I)\,(\forall\,\varepsilon < \varepsilon_0)\,(\forall\,x \geqslant -s_{m_{l(\varepsilon)}}),$$

$$\Big|v_{\varepsilon,1}^{(j)}\big(x + s_{m_{l(\varepsilon)}}\big) - v_{\varepsilon,2}^{(j)}\big(x + s_{m_{l(\varepsilon)}}\big) + w_{\varepsilon,1}^{(j)}\big(x + s_{m_{l(\varepsilon)}}\big) - w_{\varepsilon,2}^{(j)}\big(x + s_{m_{l(\varepsilon)}}\big)\Big| \leqslant c\varepsilon^m,$$

so when $l \to +\infty$ we obtain

$$(\forall m > 0)\,(\exists\,c > 0)\,(\exists\,\varepsilon_0 \in I)\,(\forall\,\varepsilon < \varepsilon_0)\,(\forall x \geqslant -s_{m_{l(\varepsilon)}}) \qquad \Big|g_{\varepsilon,1}^{(j)}(x) - g_{\varepsilon,2}^{(j)}(x)\Big| \leqslant c\varepsilon^m,$$

by taking the translate $-s_{m_{l(\varepsilon)}}$ and let $l \to +\infty$ we get

$$(\forall m > 0)\,(\exists\,c > 0)\,(\exists\,\varepsilon_0 \in I)\,(\forall\,\varepsilon < \varepsilon_0)\,(\forall\,x \geqslant 0) \qquad \Big|v_{\varepsilon,1}^{(j)}(x) - v_{\varepsilon,2}^{(j)}(x)\Big| \leqslant c\varepsilon^m.$$

By [9, Proposition 3 (5)], it follows

$$(\forall\,k \in \mathbb{Z}_+)\,(\forall\,m > 0)\,(\exists\,c > 0)\,(\exists\,\varepsilon_0 \in I)\,(\forall\,\varepsilon < \varepsilon_0) \qquad \big|v_{\varepsilon,1} - v_{\varepsilon,2}\big|_{k,\infty,\mathbb{R}} \leqslant c\varepsilon^m, \tag{25}$$

which shows that $(v_{\varepsilon,1} - v_{\varepsilon,2})_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{R})$, so $\widetilde{v}_1 = \widetilde{v}_2$ in $\mathscr{G}_{\mathscr{B}}(\mathbb{R})$. From (24) and (25) it holds that $(w_{\varepsilon,1} - w_{\varepsilon,2})_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{J})$, i. e. $\widetilde{w}_1 = \widetilde{w}_2$ on $\mathbb{J}$. $\rhd$

Notation 4. Let $\widetilde{u} \in \mathscr{G}_{aaa}$ and $\widetilde{u} = \widetilde{v} + \widetilde{w}$ on $\mathbb{J}$, where $\widetilde{v} \in \mathscr{G}_{aa}$ and $\widetilde{w} \in \mathscr{G}_{+,0}$, then $\widetilde{v}$ and $\widetilde{w}$ are called respectively the principal term and the corrective term of $\widetilde{u}$ and we denote them respectively $\widetilde{u}_{aa}$ and $\widetilde{u}_{cor}$. Also $\widetilde{u} = (\widetilde{u}_{aa} + \widetilde{u}_{cor}) \in \mathscr{G}_{aaa}$ means that $\widetilde{u}_{aa} \in \mathscr{G}_{aa}$, $\widetilde{u}_{cor} \in \mathscr{G}_{+,0}$ and $\widetilde{u} = \widetilde{u}_{aa} + \widetilde{u}_{cor}$ on $\mathbb{J}$.

6. Nonlinear Operations

The algebra of tempered generalized functions on $\mathbb{C}$ denoted by $\mathscr{G}_\tau(\mathbb{C})$, see [23] for more details, is the quotient algebra

$$\mathscr{G}_\tau(\mathbb{C}) := \frac{\mathscr{M}_\tau(\mathbb{C})}{\mathscr{N}_\tau(\mathbb{C})},$$

where

$$\mathscr{M}_\tau(\mathbb{C}) := \left\{ \begin{array}{l} (f_\varepsilon)_\varepsilon \in \left(\mathscr{E}\left(\mathbb{R}^2\right)\right)^I : \forall\,j \in \mathbb{Z}_+^2,\ \exists\,m \in \mathbb{Z}_+, \\ \sup_{x\in\mathbb{R}^2} (1 + |x|)^{-m} \left|f_\varepsilon^{(j)}(x)\right| = O\left(\varepsilon^{-m}\right),\ \varepsilon \to 0 \end{array} \right\}.$$

$$\mathscr{N}_\tau(\mathbb{C}) := \left\{ \begin{array}{l} (f_\varepsilon)_\varepsilon \in \left(\mathscr{E}\left(\mathbb{R}^2\right)\right)^I : \forall\,j \in \mathbb{Z}_+^2,\ \exists\,n \in \mathbb{Z}_+,\ \forall\,m \in \mathbb{Z}_+, \\ \sup_{x\in\mathbb{R}^2} (1 + |x|)^{-n} \left|f_\varepsilon^{(j)}(x)\right| = O\left(\varepsilon^{m}\right), \varepsilon \to 0 \end{array} \right\}.$$

Example 2. Any polynomial function is a tempered generalized function.

Theorem 4. *Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{aaa}$ and $\widetilde{F} = [(f_\varepsilon)_\varepsilon] \in \mathscr{G}_\tau(\mathbb{C})$, then*

$$\widetilde{F} \circ \widetilde{u} := \big[(f_\varepsilon \circ u_\varepsilon)_\varepsilon\big]$$

is a well-defined element of $\mathscr{G}_{aaa}$. The principal term and the corrective term of $\widetilde{F} \circ \widetilde{u}$ are respectively $\widetilde{F}(\widetilde{u}_{aa})$ and $\widetilde{F}(\widetilde{u}_{aa} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{aa})$, where $\widetilde{u} = \widetilde{u}_{aa} + \widetilde{u}_{cor}$ on $\mathbb{J}$.

$\triangleleft$ Let $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(f_\varepsilon)_\varepsilon \in \mathscr{M}_\tau(\mathbb{C})$, by the classical Faà di Bruno formula, we have $\forall j \in \mathbb{Z}_+$,

$$\frac{(f_\varepsilon \circ u_\varepsilon)^{(j)}(x)}{j!} = \sum_{\substack{l_1+2l_2+\dots+jl_j=j,\\ r=l_1+\dots+l_j}} \frac{f_\varepsilon^{(r)}(u_\varepsilon(x))}{l_1!\dots l_j!} \prod_{i=1}^{j} \left(\frac{u_\varepsilon^{(i)}(x)}{i!}\right)^{l_i}. \tag{26}$$

As $\forall\, \varepsilon \in I$, $\forall\, j \in \mathbb{Z}_+$, $u_\varepsilon^{(j)} \in \mathscr{C}_{aaa}$ and $f_\varepsilon \in \mathscr{E}(\mathbb{R}^2)$, it follows by [9, Proposition 3 (4)], that $f_\varepsilon^{(r)}(u_\varepsilon) \in \mathscr{C}_{aaa}$, and since $\mathscr{C}_{aaa}$ is an algebra, then $\forall\, \varepsilon \in I$, $f_\varepsilon \circ u_\varepsilon \in \mathscr{B}_{aaa}$. As $(u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$, then

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, n_k \in \mathbb{Z}_+)\,(\exists\, c_k > 0)\,(\exists\, \varepsilon_k \in I)\,(\forall\, \varepsilon < \varepsilon_k) \quad |u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant c_k \varepsilon^{-n_k}.$$

The fact that $(f_\varepsilon)_\varepsilon \in \mathscr{M}_\tau(\mathbb{C})$ gives

$$(\forall\, j \in \mathbb{Z}_+)\,(\exists\, N_j \in \mathbb{Z}_+)\,(\exists\, C_j > 0)\,(\exists\, \varepsilon'_j \in I)\,(\forall\, \varepsilon < \varepsilon'_j)$$
$$\left\|f_\varepsilon^{(j)}(u_\varepsilon)\right\|_{L^\infty(\mathbb{R})} \leqslant C_j \varepsilon^{-N_j} \left\|1+u_\varepsilon\right\|_{L^\infty(\mathbb{R})}^{N_j}.$$

Consequently, by (26) we obtain

$$\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} \leqslant \sum_{\substack{l_1+2l_2+\dots+jl_j=j,\\ r=l_1+\dots+l_j}} \frac{C_r \varepsilon^{-N_r} \left\|1+u_\varepsilon\right\|_{L^\infty(\mathbb{R})}^{N_r}}{l_1!\dots l_j!} \prod_{i=1}^{j} \left(\frac{\left\|u_\varepsilon^{(i)}\right\|_{L^\infty(\mathbb{R})}}{i!}\right)^{l_i},$$

hence there exists $c > 0$,

$$\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} \leqslant \sum_{\substack{l_1+2l_2+\dots+jl_j=j,\\ r=l_1+\dots+l_j}} \frac{c\varepsilon^{-N_r(1+n_0)}}{l_1!\cdots l_j!} \prod_{i=1}^{j} \left(\frac{c_i \varepsilon^{-n_i}}{i!}\right)^{l_i}$$
$$\leqslant \sum_{\substack{l_1+2l_2+\dots+jl_j=j,\\ r=l_1+\dots+l_j}} \frac{c\varepsilon^{-\left(N_r(1+n_0)+\sum_{i=1}^{j} n_i l_i\right)}}{l_1!\dots l_j!} \prod_{i=1}^{j} \left(\frac{c_i}{i!}\right)^{l_i} \leqslant C' \varepsilon^{-m},$$

where

$$m = \max_{\substack{l_1+2l_2+\dots+jl_j=j,\\ r=l_1+\dots+l_j,\\ 1\leqslant r\leqslant j}} \left\{N_r(1+n_0) + \sum_{i=1}^{j} n_i l_i\right\}, \quad C' = \sum_{\substack{l_1+2l_2+\dots+jl_j=j,\\ r=l_1+\dots+l_j}} \frac{c}{l_1!\dots l_j!} \prod_{i=1}^{j} \left(\frac{c_i}{i!}\right)^{l_i}.$$

Finally, with $C = C' \sum_{j\leqslant k} j!$, it holds

$$(\forall\, k \in \mathbb{Z}_+)\,(\exists\, m \in \mathbb{Z}_+)\,(\exists\, C > 0)\,\big(\exists\, \varepsilon'' = \inf_{1\leqslant i\leqslant j\leqslant k} (\varepsilon_i, \varepsilon'_j)\big)\,(\forall\, \varepsilon < \varepsilon'') \quad |f_\varepsilon \circ u_\varepsilon|_{k,\infty,\mathbb{R}} \leqslant C\varepsilon^{-m},$$

which means that $(f_\varepsilon \circ u_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$. This composition does not depend on the representatives. Indeed, suppose that $(v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$ and $(g_\varepsilon)_\varepsilon \in \mathscr{M}_\tau(\mathbb{C})$ are others representatives of $\widetilde{u}$ and $\widetilde{F}$ respectively. Set $(n_\varepsilon)_\varepsilon := ((v_\varepsilon)_\varepsilon - (u_\varepsilon)_\varepsilon) \in \mathscr{N}_{aaa}$ and $(m_\varepsilon)_\varepsilon := ((f_\varepsilon)_\varepsilon - (g_\varepsilon)_\varepsilon) \in \mathscr{N}_\tau(\mathbb{C})$. To show that $(f_\varepsilon \circ u_\varepsilon - g_\varepsilon \circ v_\varepsilon)_\varepsilon \in \mathscr{N}_{aaa}$, since $(f_\varepsilon \circ u_\varepsilon - g_\varepsilon \circ v_\varepsilon)_\varepsilon \in \mathscr{M}_{aaa}$, according to Proposition 3 (1), it is enough to prove that $(f_\varepsilon \circ u_\varepsilon - g_\varepsilon \circ v_\varepsilon)_\varepsilon$ satisfies (13). Indeed, we have

$$\left|f_\varepsilon \circ u_\varepsilon - g_\varepsilon \circ v_\varepsilon\right|_{0,\infty,\mathbb{R}} \leqslant \left|f_\varepsilon \circ u_\varepsilon - f_\varepsilon \circ v_\varepsilon\right|_{0,\infty,\mathbb{R}} + \left|f_\varepsilon \circ v_\varepsilon - g_\varepsilon \circ v_\varepsilon\right|_{0,\infty,\mathbb{R}}$$
$$= \left|f_\varepsilon(u_\varepsilon) - f_\varepsilon(u_\varepsilon + n_\varepsilon)\right|_{0,\infty,\mathbb{R}} + \left|m_\varepsilon(v_\varepsilon)\right|_{0,\infty,\mathbb{R}} \leqslant \left|n_\varepsilon\right|_{0,\infty,\mathbb{R}} \left|f'_\varepsilon(u_\varepsilon)\right|_{0,\infty,\mathbb{R}} + \left|m_\varepsilon(v_\varepsilon)\right|_{0,\infty,\mathbb{R}}.$$

It is clear that

$$(\forall\, k \in \mathbb{Z}_+) \quad \left|n_\varepsilon\right|_{0,\infty,\mathbb{R}} \left|f'_\varepsilon(u_\varepsilon)\right|_{0,\infty,\mathbb{R}} = O\big(\varepsilon^k\big),\ \varepsilon \to 0,$$

and also

$$(\forall\, l \in \mathbb{Z}_+) \quad \left|m_\varepsilon(v_\varepsilon)\right|_{0,\infty,\mathbb{R}} = O\big(\varepsilon^l\big),\ \varepsilon \to 0.$$

Therefore,

$$(\forall\, q \in \mathbb{Z}_+) \quad \left|f_\varepsilon \circ u_\varepsilon - g_\varepsilon \circ v_\varepsilon\right|_{0,\infty,\mathbb{R}} = O\big(\varepsilon^q\big),\ \varepsilon \to 0.$$

Let $\widetilde{u} = (\widetilde{u}_{aa} + \widetilde{u}_{cor}) \in \mathscr{G}_{aaa}$. As $\widetilde{F} \circ \widetilde{u} = \widetilde{F}(\widetilde{u}_{aa}) + (\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{u}_{aa}))$, then

$$\widetilde{F} \circ \widetilde{u} = \widetilde{F}(\widetilde{u}_{aa}) + \big(\widetilde{F}(\widetilde{u}_{aa} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{aa})\big) \quad \text{on} \quad \mathbb{J}.$$

In view of [22, Proposition 9], we obtain $\widetilde{F}(\widetilde{u}_{aa}) \in \mathscr{G}_{aa}$. It remains to prove that $\widetilde{F}(\widetilde{u}_{aa}+\widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{aa}) \in \mathscr{G}_{+,0}$. Since $\mathscr{G}_{aaa}$ and $\mathscr{G}_{aa}$ are contained in $\mathscr{G}_{\mathscr{B}}$ then $\widetilde{F}(\widetilde{u}_{aa} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{aa}) \in \mathscr{G}_{\mathscr{B}}$. It suffices to show that

$$(\forall\, \varepsilon \in I) \quad f_\varepsilon\left(u_{aa,\varepsilon} + u_{cor,\varepsilon}\right) - f_\varepsilon\left(u_{aa,\varepsilon}\right) \in \mathscr{B}_{+,0},$$

where $(f_\varepsilon)_\varepsilon, (u_{aa,\varepsilon})_\varepsilon$ and $(u_{cor,\varepsilon})_\varepsilon$ are respective representatives of $\widetilde{F}, \widetilde{u}_{aa}$ and $\widetilde{u}_{cor}$. The classical result on composition of asymptotically almost automorphic function with continuous function shows that the corrective term of $f_\varepsilon(u_{aa,\varepsilon} + u_{cor,\varepsilon})$ is $f_\varepsilon(u_{aa,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{aa,\varepsilon})$ and the fact that $\widetilde{F}(\widetilde{u}_{aa} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{aa}) \in \mathscr{G}_{\mathscr{B}}$ gives

$$(\forall\, \varepsilon \in I) \quad (f_\varepsilon(u_{aa,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{aa,\varepsilon})) \in \mathscr{B}.$$

By [9, Proposition 5 (5)], we have

$$(\forall\, \varepsilon \in I) \quad f_\varepsilon(u_{aa,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{aa,\varepsilon}) \in \mathscr{C}_{+,0} \cap \mathscr{B} = \mathscr{B}_{+,0}. \quad \triangleright$$

7. Linear Neutral Difference Differential Systems

We consider linear neutral difference differential systems for the unknown vector function $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)$,

$$L_\omega \widetilde{u} := \sum_{i=0}^{p} \sum_{j=0}^{q} \widetilde{A}_{ij}\left(\tau_{\omega_j}\widetilde{u}\right)^{(i)} + \widetilde{K} * \widetilde{u} = \widetilde{f}, \tag{27}$$

where $\widetilde{f} = \big(\widetilde{f}_1, \ldots, \widetilde{f}_n\big) \in (\mathscr{G}_{aaa})^n$, $\omega = (\omega_j)_{0\leqslant j \leqslant q} \subset \mathbb{R}_+^q$ and for $i \leqslant p, j \leqslant q$, $\widetilde{A}_{ij} = \big(\widetilde{A}_{ij}^{rl}\big)_{1\leqslant r,l\leqslant n}$ and $\widetilde{K} = \big(\widetilde{K}^{rl}\big)_{1\leqslant r,l\leqslant n}$ are square matrices of almost automorphic generalized functions and L^1 — generalized functions respectively.

Lemma 2. *If $\widetilde{u} \in (\mathscr{G}_{aaa})^n$ then $L_\omega \widetilde{u} \in (\mathscr{G}_{aaa})^n$.*

$\triangleleft$ If $\widetilde{u} \in (\mathscr{G}_{aaa})^n$ then due to results of Proposition 5 we obtain that

$$(\forall\, i \leqslant p, j \leqslant q)\left(\forall\, \omega = (\omega_j)_{0\leqslant j\leqslant q} \subset \mathbb{R}_+^q\right) \quad \left(\widetilde{A}_{ij}\left(\tau_{\omega_j}\widetilde{u}\right)^{(i)}\right) \in (\mathscr{G}_{aaa})^n,$$

and also

$$\widetilde{K} * \widetilde{u} \in (\mathscr{G}_{aaa})^n,$$

so

$$\left(\sum_{i=0}^{p}\sum_{j=0}^{q}\widetilde{A}_{ij}\left(\tau_{\omega_j}\widetilde{u}\right)^{(i)}+\widetilde{K}*\widetilde{u}\right)\in\left(\mathscr{G}_{aaa}\right)^n,$$

i. e. $L_\omega\widetilde{u}\in(\mathscr{G}_{aaa})^n$. $\triangleright$

Definition 4. A generalized function $\widetilde{u}\in(\mathscr{G}_{\mathscr{B}})^n$ is said a generalized solution of (27) on $\mathbb{J}$ if it satisfies

$$\left(\sum_{l=1}^{n}\sum_{i=1}^{p}\sum_{j=1}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{l,\varepsilon}\right)^{(i)}+K_\varepsilon^{rl}*u_{l,\varepsilon}-f_{r,\varepsilon}\right)_\varepsilon\in\mathscr{N}_{\mathscr{B}}(\mathbb{J}),\quad r=1,\dots,n,$$

where $(u_{l,\varepsilon})_\varepsilon$, $\left(A_{ij,\varepsilon}^{rl}\right)_\varepsilon$, $\left(K_\varepsilon^{rl}\right)_\varepsilon$ and $(f_{r,\varepsilon})_\varepsilon$, $r,l=1,\dots,n$, are respective representatives of $\widetilde{u}_r$, $\widetilde{A}_{ij}^{rl}$, $\widetilde{K}^{rl}$ and $\widetilde{f}_r$.

We give the main result of this section.

Theorem 5. *Let $\widetilde{f}=(\widetilde{f}_{aa}+\widetilde{f}_{cor})\in(\mathscr{G}_{aaa})^n$, the equation (27) admits a generalized solution $\widetilde{u}\in(\mathscr{G}_{aaa})^n$ on $\mathbb{J}$ if and only if there exist $\widetilde{v}\in(\mathscr{G}_{aa})^n$ and $\widetilde{w}\in(\mathscr{G}_{+,0})^n$ such that*

$$L_\omega\widetilde{v}=\widetilde{f}_{aa}\quad\text{on}\quad\mathbb{R},\tag{28}$$

and

$$L_\omega\widetilde{w}=\widetilde{f}_{cor}\quad\text{on}\quad\mathbb{J}.\tag{29}$$

$\triangleleft$ If $\widetilde{u}=(\widetilde{u}_{aa}+\widetilde{u}_{cor})\in(\mathscr{G}_{aaa})^n$ is a generalized solution of (27), then the equations (27) explicitly are written as follows

$$\begin{cases}\sum\limits_{l=1}^{n}\left(\sum\limits_{i=0}^{p}\sum\limits_{j=0}^{q}\widetilde{A}_{ij}^{1l}\left(\tau_{\omega_j}\widetilde{u}_l\right)^{(i)}+\widetilde{K}^{1l}*\widetilde{u}_l\right)=\widetilde{f}_1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \sum\limits_{l=1}^{n}\left(\sum\limits_{i=0}^{p}\sum\limits_{j=0}^{q}\widetilde{A}_{ij}^{nl}\left(\tau_{\omega_j}\widetilde{u}_l\right)^{(i)}+\widetilde{K}^{nl}*\widetilde{u}_l\right)=\widetilde{f}_n.\end{cases}$$

Let $(u_{aa,l,\varepsilon})_\varepsilon,(u_{cor,l,\varepsilon})_\varepsilon$, $\left(A_{ij,\varepsilon}^{rl}\right)_\varepsilon$, $\left(K_\varepsilon^{rl}\right)_\varepsilon$, $(f_{aa,l,\varepsilon})_\varepsilon$ and $(f_{cor,l,\varepsilon})_\varepsilon$ be respective representatives of $\widetilde{u}_{aa,l}$, $\widetilde{u}_{cor,l}$, $\widetilde{A}_{ij}^{rl}$, $\widetilde{K}^{rl}$, $\widetilde{f}_{aa,l}$ and $\widetilde{f}_{cor,l}$, $1\leqslant l\leqslant n$. For every $r=1,\dots,n$ and $\varepsilon\in I$, denoting

$$S_{aa,r,\varepsilon}:=\sum_{l=1}^{n}\left(\sum_{i=1}^{p}\sum_{j=1}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{aa,l,\varepsilon}\right)^{(i)}+K_\varepsilon^{rl}*u_{aa,l,\varepsilon}\right),$$

$$S_{cor,r,\varepsilon}:=\sum_{l=1}^{n}\left(\sum_{i=1}^{p}\sum_{j=1}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{cor,l,\varepsilon}\right)^{(i)}+K_\varepsilon^{rl}*u_{cor,l,\varepsilon}\right),$$

we have $\forall\varepsilon\in I$, $S_{aa,r,\varepsilon}\in\mathscr{B}_{aa}$ and $S_{cor,r,\varepsilon}\in\mathscr{B}_{+,0}$, and also $[(S_{aa,r,\varepsilon})_\varepsilon]\in\mathscr{G}_{aa}$ and $[(S_{cor,r,\varepsilon})_\varepsilon]\in\mathscr{G}_{+,0}$. Since $\widetilde{u}$ is a generalized solution of (27) on $\mathbb{J}$, it means that

$$(\forall\,k\in\mathbb{Z}_+)\,(\forall\,m>0)\,(\exists\,c_k>0)\,(\exists\,\varepsilon_k\in I)\,(\forall\,\varepsilon<\varepsilon_k)\,(\forall\,x\in\mathbb{J})$$

$$\sum_{j\leqslant k}\left|S_{aa,r,\varepsilon}^{(j)}(x)-f_{aa,r,\varepsilon}^{(j)}(x)+S_{cor,r,\varepsilon}^{(j)}(x)-f_{cor,r,\varepsilon}^{(j)}(x)\right|\leqslant c_k\varepsilon^m,\quad r=1,\dots,n.\tag{30}$$

For any real sequence $(s_m)_{m\in\mathbb{N}}$, such that $s_m \to +\infty$ there exist $(s_{m_{p(\varepsilon)}})_p$ a subsequence of $(s_m)_{m\in\mathbb{N}}$, such that taking the translate at $s_{m_{p(\varepsilon)}} - s_{m_{q(\varepsilon)}}$ in the estimate (30) and let $p, q \to +\infty$, we obtain due to [9, Proposition 9], that

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m > 0)\,(\exists\, c_k > 0)\,(\exists\, \varepsilon_k \in I)\,(\forall\, \varepsilon < \varepsilon_k)\,(\forall\, x \in \mathbb{J})$$

$$\sum_{j\leqslant k} \left|S^{(j)}_{aa,r,\varepsilon}(x) - f^{(j)}_{aa,r,\varepsilon}(x)\right| \leqslant c_k\varepsilon^m \quad (\forall\, r = 1, \ldots, n),$$

consequently, by [9, Proposition 3 (5)], it holds

$$(\forall\, k \in \mathbb{Z}_+)\,(\forall\, m > 0)\,(\exists\, c_k > 0)\,(\exists\, \varepsilon_k \in I)\,(\forall\, \varepsilon < \varepsilon_k)$$

$$\left|S_{aa,r,\varepsilon} - f_{aa,r,\varepsilon}\right|_{k,\infty,\mathbb{R}} \leqslant c_k\varepsilon^m \quad (\forall\, r = 1, \ldots, n),$$

i. e.

$$\left(S_{aa,r,\varepsilon} - f_{aa,r,\varepsilon}\right)_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{R}) \quad (\forall r = 1, \ldots, n), \tag{31}$$

which means that $\widetilde{u}_{aa} = (\widetilde{u}_{aa,1}, \ldots, \widetilde{u}_{aa,n})$ is a generalized solution of (28) on $\mathbb{R}$. By (30) and (31), we deduce

$$\left(S_{cor,r,\varepsilon} - f_{cor,r,\varepsilon}\right)_\varepsilon \in \mathscr{N}_{\mathscr{B}}(\mathbb{J}) \quad (\forall r = 1, \ldots, n),$$

i.e. $\widetilde{u}_{cor} = (\widetilde{u}_{cor,1}, \ldots, \widetilde{u}_{cor,n})$ is a generalized solution of (29) on $\mathbb{J}$.

Conversely if there exist $\widetilde{v} \in (\mathscr{G}_{aa})^n$ and $\widetilde{w} \in (\mathscr{G}_{+,0})^n$ such that (28) and (29) hold, then we have $\widetilde{u} := (\widetilde{v} + \widetilde{w}) \in (\mathscr{G}_{aaa})^n$ is a generalized solution of (27) on $\mathbb{J}$. ▷

Remark 3. Theorem 5 generalizes Theorem 6 of [21] and Theorem 3 of [9].

As a particular case we consider linear systems of ordinary differential equations

$$\dot{\widetilde{u}} + \widetilde{A}\widetilde{u} = \widetilde{f}, \tag{32}$$

where $\widetilde{A}$ is a square matrix of almost automorphic generalized functions.

Corollary 1. *Let $\widetilde{f} = \left(\widetilde{f}_{aa} + \widetilde{f}_{cor}\right) \in (\mathscr{G}_{aaa})^n$, the system (32) admits a generalized solution $\widetilde{u} \in (\mathscr{G}_{aaa})^n$ on $\mathbb{J}$ if and only if there exist $\widetilde{v} \in (\mathscr{G}_{aa})^n$ and $\widetilde{w} \in (\mathscr{G}_{+,0})^n$ such that*

$$\dot{\widetilde{v}} + \widetilde{A}\widetilde{v} = \widetilde{f}_{aa} \quad \text{on} \quad \mathbb{R},$$

and

$$\dot{\widetilde{w}} + \widetilde{A}\widetilde{w} = \widetilde{f}_{cor} \quad \text{on} \quad \mathbb{J}.$$

Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{B}}$, $x_0 \in \mathbb{R}$ and define the primitive of $\widetilde{u}$ by $\widetilde{U} = [(U_\varepsilon)_\varepsilon]$, where

$$U_\varepsilon(x) := \int\limits_{x_0}^{x} u_\varepsilon(t)\,dt, \quad \varepsilon \in I.$$

Corollary 2. *A generalized function $\widetilde{U} = \left(\widetilde{U}_{aa} + \widetilde{U}_{cor}\right) \in \mathscr{G}_{\mathscr{B}}$ is a primitive of $\widetilde{u} = (\widetilde{u}_{aa} + \widetilde{u}_{cor}) \in \mathscr{G}_{aaa}$ on $\mathbb{J}$ if and only if*

$$\widetilde{U}_{aa} \ \text{ is a primtive of } \ \widetilde{u}_{aa} \ \text{ on } \mathbb{R},$$

and

$$\widetilde{U}_{cor} \ \text{ is a primitive of } \ \widetilde{u}_{cor} \ \text{ on } \mathbb{J}.$$

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АСИМПТОТИЧЕСКАЯ ПОЧТИ АВТОМОРФНОСТЬ ДЛЯ АЛГЕБР ОБОБЩЕННЫХ ФУНКЦИЙ

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Аннотация. Цель статьи — изучение понятие асимптотической почти автоморфности в контексте обобщенных функций. Вводится алгебра асимптотически почти автоморфных обобщенных функций, содержащих пространство гладких асимптотически почти автоморфных функции как подалгебру. Фундаментальное значение этой алгебры связана с невозможностью умножения распределений; оно также содержит асимптотически почти автоморфные распределения Соболева — Шварца как подпространство. Более того, показано, что введенная алгебра устойчива относительно некоторых нелинейных операций. Как побочного результата приводится результат типа Сили о продолжении функций в контекст алгебры ограниченных обобщенных функций и алгебры ограниченных обобщенных функций, обращающихся в нуль на бесконечности; эти результаты используется для доказательства фундаментального результата о единственности разложения асимптотически почти автоморфной обобщенной функции. В качестве приложений рассмотрены разностно-дифференциальные системы нейтрального типа в рамках изучаемой алгебра обобщенных функций.

Ключевые слова: асимптотическая почти автоморфность, обобщенные функции, нейтральные дифференциально-разностные уравнения.

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