# ON GENERALIZED KENMOTSU MANIFOLDS AS HYPERSURFACES OF VAISMAN–GRAY MANIFOLDS

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**Abstract.** In this paper, we conclude that the hypersurfaces of Vaisman–Gray manifolds have generalized Kenmotsu structures under some conditions for the Lee form, Kirichenko's tensors and the second fundamental form of the immersion of the hypersurface into the manifold of Vaisman–Gray class. Moreover, the components of the second fundamental form are determined when the foregoing hypersurfaces have generalized Kenmotsu structures or any special kind of it or Kenmotsu structures, such that some of these components are vanish. Also, some components of Lee form and some components of some Kirichenko's tensors in the Vaisman–Gray class are equal to zero. On the other hand, the minimality of totally umbilical, totally geodesic hypersurfaces of Vaisman–Gray manifolds with generalized Kenmotsu structures are investigated. In addition, we deduced that the hypersurface of Vaisman–Gray manifold that have generalized Kenmotsu structure is totally geodesic if and only if it is totally umbilical and some components of Lee form are constants.

**Keywords:** almost contact metric manifolds, nearly Kenmotsu manifolds, Vaisman–Gray manifolds, hypersurfaces of almost Hermitian manifolds.

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## 1. Introduction

In 1972, K. Kenmotsu introduced a new class of almost contact metric manifolds [1]. In 2002, S. V. Umnova [2] introduced a generalization of Kenmotsu manifolds that studied later by I. K. Erken et al. [3] and N. Heidari et al. [4] under the title nearly Kenmotsu manifolds. Whereas, A. Abu-Saleem and A. R. Rustanov [5] investigated some curvature identities of a special subclass of that class introduced by S. V. Umnova [2]. Additionally, H. M. Abood and M. Y. Abass [6] constructed a new manifold which is a generalization of Kenmotsu manifold and they called it a manifold of Kenmotsu type.

On the other hand, M. B. Banaru and V. F. Kirichenko [7] focused on the hypersurfaces of almost Hermitian manifolds that have almost contact metric structures. They also derived the complete first group of Cartan's structure equations. Continued in the same way, L. V. Stepanova et al. [8] discussed the hypersurfaces of the Kählerian manifolds that have quasi-Sasakian structure. Moreover, M. B. Banaru and G. A. Banaru [9] demonstrated that a minimal hypersurfaces of a Kählerian manifold, equipped with an almost contact metric Kirichenko–Uskorev structure that found in [10], is totally umbilical if and only if it is totally

geodesic. Also, M. B. Banaru et al. [11] showed that 2-hypersurfaces and 3-hypersurfaces of the almost Hermitian manifolds of class $W_4$ admit almost contact metric structures.

Therefore, in this article, we studied the hypersurfaces of Vaisman–Gray manifold which equipped with generalized Kenmotsu structure in Section 3, after the preliminaries in Section 2.

## 2. Preliminaries

We symbolize by $M^n$ a smooth manifold $M$ of dimension $n$ and $X(M)$ is the Lie algebra of smooth vector fields over $M$.

DEFINITION 1 [12]. An almost contact metric manifold (shortly, $ACM$-manifold) is the smooth manifold $M^{2n+1}$ equipped with the quadruple $(\Phi, \xi, \eta, g)$, where $\Phi$ is a $(1,1)$-tensor, $\xi \in X(M)$, $\eta$ is the dual of $\xi$, and $g$ is the Riemannian metric such that the following conditions hold:

$$\Phi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -\mathrm{id} + \eta \otimes \xi,$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (\forall\, X, Y \in X(M)).$$

DEFINITION 2 [13]. An almost Hermitian manifold (shortly, $AH$-manifold) is the smooth manifold $N^{2m}$ equipped with the pair $(J, h)$, where $J$ is a $(1, 1)$-tensor and $h$ is the Riemannian metric such that the following hold:

$$J^2 = -\mathrm{id}, \quad h(JX, JY) = h(X, Y) \quad (\forall\, X, Y \in X(N)).$$

DEFINITION 3 [14]. A generalized Kenmotsu manifold (shortly, $GK$-manifold) is an $ACM$-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ which satisfies the following identity:

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = -\eta(Y)\Phi X - \eta(X)\Phi Y \quad (\forall\, X, Y \in X(M)),$$

where $\nabla$ is the Riemannian connection (i. e. Levi-Civita connection) of the metric $g$ on $M$. On the other hand, the $ACM$-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ becomes Kenmotsu manifold if the above identity reduces to the following:

$$\nabla_X(\Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X \quad (\forall\, X, Y \in X(M)).$$

DEFINITION 4 [13]. A Vaisman–Gray manifold (shortly, $VG$-manifold) is an $AH$-manifold $(N^{2m}, J, h)$ which satisfies the following identity:

$$\widetilde{\nabla}_X(F)(X, Y) = -\frac{1}{2(m-1)}\left\{ h(X, X)\delta F(Y) - h(X, Y)\delta F(X) - h(JX, Y)\delta F(JX) \right\}$$

$$(\forall\, X, Y \in X(N)),$$

where $F(X, Y) = h(JX, Y)$, $\delta$ is the codifferential operator and $\widetilde{\nabla}$ is the Riemannian connection (i. e. Levi-Civita connection) on $N$.

REMARK 1. The method of associated $G$-structure (briefly, $AG$-structure) space introduced by V. F. Kirichenko [15] and used by A. R. Rustanov et al. [16] in Russian version and exists in [5] as an English translation. So, the authors can learn more about this method from these citations. Also, we summarize the $AG$-structure space of $ACM$-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ as follow:

Suppose that $\{\xi = e_0, e_1, \ldots, e_n, \Phi(e_1), \ldots, \Phi(e_n)\}$ is an orthonormal basis of $X(M)$, then Kirichenko's method produced another basis as

$$\{\xi = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}\},$$

where

$$\varepsilon_a = \frac{1}{\sqrt{2}}\left(e_a - \sqrt{-1}\Phi e_a\right), \quad \varepsilon_{\hat{a}} = \frac{1}{\sqrt{2}}\left(e_a + \sqrt{-1}\Phi e_a\right), \quad a = 1, 2, \ldots, n, \quad \hat{a} = a + n.$$

So, we get:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & O & I_n \\ 0 & I_n & O \end{pmatrix}, \qquad (\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}\,I_n & O \\ 0 & O & -\sqrt{-1}\,I_n \end{pmatrix}, \tag{1}$$

where $I_n$ is $n \times n$ identity matrix and $i, j = 0, 1, \ldots, 2n$.

On the other hand, the $AG$-structure space of $AH$-manifold $(N^{2m}, J, h)$ is summarized by [15]:

Suppose that $\{\tilde{e}_1, \ldots, \tilde{e}_m, J(\tilde{e}_1), \ldots, J(\tilde{e}_m)\}$ is an orthonormal basis of $X(N)$, then Kirichenko's method produced another basis as $\{\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_m, \tilde{\varepsilon}_{\hat{1}}, \ldots, \tilde{\varepsilon}_{\hat{m}}\}$, where

$$\tilde{\varepsilon}_b = \frac{1}{\sqrt{2}}\left(\tilde{e}_b - \sqrt{-1}J\tilde{e}_b\right), \quad \tilde{\varepsilon}_{\hat{b}} = \frac{1}{\sqrt{2}}\left(\tilde{e}_b + \sqrt{-1}J\tilde{e}_b\right).$$

So, we get:

$$(h_{kl}) = \begin{pmatrix} O & I_m \\ I_m & O \end{pmatrix}, \qquad (J_l^k) = \begin{pmatrix} \sqrt{-1}\,I_m & O \\ O & -\sqrt{-1}\,I_m \end{pmatrix}, \tag{2}$$

where $b = 1, 2, \ldots, m$, $\hat{b} = b + m$, and $k, l = 1, 2, \ldots, 2m$.

**Theorem 1** [14]. *The first group of Cartan's structure equations of $GK$-manifold of dimension $2n + 1$ on the $AG$-structure space is given as the following:*

1) $d\omega = F_{ab}\,\omega^a \wedge \omega^b + F^{ab}\,\omega_a \wedge \omega_b$;
2) $d\omega^a = \omega_b^a \wedge \omega^b + C^{abc}\,\omega_b \wedge \omega_c + \frac{3}{2}\,F^{ab}\,\omega_b \wedge \omega - \delta_b^a \omega^b \wedge \omega$;
3) $d\omega_a = -\omega_a^b \wedge \omega_b + C_{abc}\,\omega^b \wedge \omega^c + \frac{3}{2}\,F_{ab}\,\omega^b \wedge \omega - \delta_a^b \omega_b \wedge \omega$, *where*

$$F^{ab} = \sqrt{-1}\,\Phi_{\hat{a},\hat{b}}^0, \quad F^{ab} + F^{ba} = 0, \quad C^{abc} = \frac{\sqrt{-1}}{2}\,\Phi_{\hat{b},\hat{c}}^a,$$

$$F_{ab} = -\sqrt{-1}\,\Phi_{a,b}^0, \quad F_{ab} + F_{ba} = 0, \quad C_{abc} = -\frac{\sqrt{-1}}{2}\,\Phi_{b,c}^{\hat{a}},$$

$$C^{[abc]} = C^{abc}, \quad \overline{C^{abc}} = C_{abc}, \quad C_{[abc]} = C_{abc}, \quad \overline{F^{ab}} = F_{ab},$$

$\Phi_{j,k}^i$ *are the components of $\nabla_X(\Phi)Y$ on the $AG$-structure space, $[\ldots]$ is the anti-symmetric operator of their indices, $a, b, c = 1, 2, \ldots, n$, $\hat{a} = a + n$ and $\overline{C^{abc}}$, $\overline{F^{ab}}$ are the complex conjugate of $C^{abc}$, $F^{ab}$ respectively.*

DEFINITION 5 [17]. The $GK$-manifold is called

1) a special generalized Kenmotsu manifold of the first kind (briefly, $SGK_1$-manifold) if $C^{abc} = C_{abc} = 0$;

2) a special generalized Kenmotsu manifold of the second kind (briefly, $SGK_2$-manifold) if $F^{ab} = F_{ab} = 0$;

3) a Kenmotsu manifold if $C^{abc} = C_{abc} = 0$ and $F^{ab} = F_{ab} = 0$.

**Theorem 2** [18]. *The first group of Cartan's structure equations of VG-manifold of dimension $2m$ on the AG-structure space is given as the following:*

1) $d\omega^a = \omega_b^a \wedge \omega^b + B_c^{ab} \omega^c \wedge \omega_b + B^{abc} \omega_b \wedge \omega_c;$

2) $d\omega_a = -\omega_a^b \wedge \omega_b + B_{ab}^c \omega_c \wedge \omega^b + B_{abc} \omega^b \wedge \omega^c,$ *where*

$$B^{abc} = \frac{\sqrt{-1}}{2} J_{[\hat{b},\hat{c}]}^a, \quad B^{[abc]} = B^{abc}, \quad \overline{B^{abc}} = B_{abc}, \quad B_{abc} = -\frac{\sqrt{-1}}{2} J_{[b,c]}^{\hat{a}},$$

$$B_{[abc]} = B_{abc}, \quad B_{ab}^c = \alpha_{[a}\delta_{b]}^c, \quad B_c^{ab} = \alpha^{[a}\delta_c^{b]}, \quad \overline{B_c^{ab}} = B_{ab}^c,$$

$\alpha$ *is the Lee form, $J_{j,k}^i$ are the components of $\widetilde{\nabla}_X(J)Y$ on the AG-structure space and $a, b, c = 1, 2, \ldots, m$, $\hat{a} = a + m$.*

**Theorem 3** [7]. *An ACM-manifold $(M^{2n-1}, \Phi, \xi, \eta, g)$ which is a hypersurface of an AH-manifold $(N^{2n}, J, h)$ has the following first family of Cartan's structure equations:*

$$d\omega^{\tilde{a}} = \omega_{\tilde{b}}^{\tilde{a}} \wedge \omega^{\tilde{b}} + B_{\tilde{c}}^{\tilde{a}\tilde{b}} \omega^{\tilde{c}} \wedge \omega_{\tilde{b}} + B^{\tilde{a}\tilde{b}\tilde{c}} \omega_{\tilde{b}} \wedge \omega_{\tilde{c}} + \left(\sqrt{2} B_{\tilde{b}}^{\tilde{a}n} + \sqrt{-1} \sigma_{\tilde{b}}^{\tilde{a}}\right) \omega^{\tilde{b}} \wedge \omega$$
$$+ \left(\sqrt{-1} \sigma^{\tilde{a}\tilde{b}} - \sqrt{2} \tilde{B}^{n\tilde{a}\tilde{b}} - \frac{1}{\sqrt{2}} B_n^{\tilde{a}\tilde{b}} - \frac{1}{\sqrt{2}} \tilde{B}^{\tilde{a}\tilde{b}n}\right) \omega_{\tilde{b}} \wedge \omega,$$

$$d\omega_{\tilde{a}} = -\omega_{\tilde{a}}^{\tilde{b}} \wedge \omega_{\tilde{b}} + B_{\tilde{a}\tilde{b}}^{\tilde{c}} \omega_{\tilde{c}} \wedge \omega^{\tilde{b}} + B_{\tilde{a}\tilde{b}\tilde{c}} \omega^{\tilde{b}} \wedge \omega^{\tilde{c}} + \left(\sqrt{2} B_{\tilde{a}n}^{\tilde{b}} - \sqrt{-1} \sigma_{\tilde{a}}^{\tilde{b}}\right) \omega_{\tilde{b}} \wedge \omega$$
$$- \left(\sqrt{-1} \sigma_{\tilde{a}\tilde{b}} + \sqrt{2} \tilde{B}_{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}} B_{\tilde{a}\tilde{b}}^n + \frac{1}{\sqrt{2}} \tilde{B}_{\tilde{a}\tilde{b}n}\right) \omega^{\tilde{b}} \wedge \omega,$$

$$d\omega = \sqrt{2} B_{n\tilde{a}\tilde{b}} \omega^{\tilde{a}} \wedge \omega^{\tilde{b}} + \sqrt{2} B^{n\tilde{a}\tilde{b}} \omega_{\tilde{a}} \wedge \omega_{\tilde{b}} + \left(\sqrt{2} B_{\tilde{b}}^{n\tilde{a}} - \sqrt{2} B_{n\tilde{b}}^{\tilde{a}} - 2\sqrt{-1}\sigma_{\tilde{b}}^{\tilde{a}}\right) \omega^{\tilde{b}} \wedge \omega_{\tilde{a}}$$
$$+ \left(\tilde{B}_{n\tilde{b}n} + B_{n\tilde{b}}^n + \sqrt{-1} \sigma_{n\tilde{b}}\right) \omega \wedge \omega^{\tilde{b}} + \left(\tilde{B}^{n\tilde{b}n} + B_n^{n\tilde{b}} - \sqrt{-1} \sigma_n^{\tilde{b}}\right) \omega \wedge \omega_{\tilde{b}},$$

*where*

$$B_c^{[ab]} = B_c^{ab}, \quad B_{[ab]}^c = B_{ab}^c, \quad \tilde{B}^{abc} = \frac{\sqrt{-1}}{2} J_{\hat{b},\hat{c}}^a, \quad \tilde{B}_{abc} = -\frac{\sqrt{-1}}{2} J_{b,c}^{\hat{a}},$$

$$B_c^{ab} = -\frac{\sqrt{-1}}{2} J_{\hat{b},c}^a, \quad B_{ab}^c = \frac{\sqrt{-1}}{2} J_{b,\hat{c}}^{\hat{a}}, \quad \overline{B_c^{ab}} = B_{ab}^c, \quad B^{abc} = \tilde{B}^{a[bc]}, \quad B_{abc} = \tilde{B}_{a[bc]},$$

*and $\sigma : X(M) \times X(M) \to X(M)$ is the second fundamental form of the immersion of the hypersurface $M^{2n-1}$ into the manifold $N^{2n}$ which is symmetric ($\sigma_{ij} = \sigma_{ji}$, $i, j = 1, 2, \ldots, 2n-1$) such that $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ with $\widetilde{\nabla}$ and $\nabla$ are the Riemannian connections of $N$ and $M$ respectively (see [19]). Further, $\tilde{a}, \tilde{b}, \tilde{c} = 1, 2, \ldots, n-1$, while $a, b, c = 1, 2, \ldots, n$ and $\omega = \omega^n = \omega_n$.*

DEFINITION 6 [20]. *A submanifold $(M^{2n-1}, g)$ of Riemannian manifold $(N^{2n}, h)$ is said to be totally umbilical if*

$$\sigma_{ij} = \lambda g_{ij},$$

*where $\sigma$ is the second fundamental form, $\lambda$ is constant and $i, j = 1, 2, \ldots, 2n-1$.*

DEFINITION 7 [21]. *A submanifold $(M^{2n-1}, g)$ of Riemannian manifold $(N^{2n}, h)$ is said to be totally geodesic if the second fundamental form $\sigma$ vanish (i. e., $\sigma = 0$).*

DEFINITION 8 [7]. *A submanifold $(M^{2n-1}, g)$ of Riemannian manifold $(N^{2n}, h)$ is said to be minimal if it satisfies $g^{ps}\sigma_{ps} = 0$, for all $p, s = 1, \ldots, 2n-1$, where $g^{ps}$ and $\sigma_{ps}$ are the components of $g^{-1}$ and the second fundamental form $\sigma$ respectively.*

### 3. $GK$-Manifolds as Hypersurfaces of $VG$-Manifolds

**Theorem 4.** *The hypersurface of a Vaisman–Gray manifold $(N^{2n}, J, h)$ admits a generalized Kenmotsu structure if and only if its second fundamental form $\sigma$ has the following components:*

$$\sigma_{\tilde{a}\tilde{b}} = \sqrt{-1}\left\{\frac{3}{2}F_{\tilde{a}\tilde{b}} + \sqrt{2}\,\tilde{B}_{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}\tilde{B}_{\tilde{a}\tilde{b}n}\right\}, \quad \sigma_{n\tilde{b}} = \sigma_n^{\tilde{b}} = 0, \quad \sigma_{\tilde{a}}^{\tilde{b}} = -\sqrt{-1}\left\{1 - \frac{1}{\sqrt{2}}\alpha_n\right\}\delta_{\tilde{a}}^{\tilde{b}},$$

$$\sigma^{\tilde{a}\tilde{b}} = -\sqrt{-1}\left\{\frac{3}{2}F^{\tilde{a}\tilde{b}} + \sqrt{2}\,\tilde{B}^{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}\tilde{B}^{\tilde{a}\tilde{b}n}\right\}, \quad \sigma_{\tilde{b}}^{\tilde{a}} = \sqrt{-1}\left\{1 - \frac{1}{\sqrt{2}}\alpha^n\right\}\delta_{\tilde{b}}^{\tilde{a}},$$

*where*

$$\alpha^{\tilde{a}} = \alpha_{\tilde{a}} = 0, \quad \alpha^n + \alpha_n = 2\sqrt{2}, \quad F^{\tilde{a}\tilde{b}} = \sqrt{2}\,B^{n\tilde{a}\tilde{b}} = -\frac{\sqrt{2}}{5}\tilde{B}^{\tilde{a}\tilde{b}n}, \quad \tilde{B}^{n\tilde{b}n} = \tilde{B}_{n\tilde{b}n} = 0,$$

$$F_{\tilde{a}\tilde{b}} = \sqrt{2}\,B_{n\tilde{a}\tilde{b}} = -\frac{\sqrt{2}}{5}\tilde{B}_{\tilde{a}\tilde{b}n}, \quad C^{\tilde{a}\tilde{b}\tilde{c}} = B^{\tilde{a}\tilde{b}\tilde{c}}, \quad C_{\tilde{a}\tilde{b}\tilde{c}} = B_{\tilde{a}\tilde{b}\tilde{c}}.$$

$\lhd$ Suppose that $N$ is $VG$-manifold, then Theorem 2 gives:

$$B^{[abc]} = B^{abc}, \quad \overline{B^{abc}} = B_{abc}, \quad B_{[abc]} = B_{abc}, \quad B_c^{ab} = \alpha^{[a}\delta_c^{b]}, \quad B_{ab}^c = \alpha_{[a}\delta_{b]}^c,$$

where $a, b, c = 1, 2, \ldots, n$. Therefore, comparing Theorem 3 with Theorem 1, we have

1) $B_{\tilde{c}}^{\tilde{a}\tilde{b}} = 0$, $B^{\tilde{a}\tilde{b}\tilde{c}} = C^{\tilde{a}\tilde{b}\tilde{c}}$, $\sigma_{\tilde{b}}^{\tilde{a}} = \sqrt{-1}\left\{\sqrt{2}\,B_{\tilde{b}}^{\tilde{a}n} + \delta_{\tilde{b}}^{\tilde{a}}\right\}$, $\sigma^{\tilde{a}\tilde{b}} = -\sqrt{-1}\left\{\sqrt{2}\,\tilde{B}^{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}B_n^{\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}\tilde{B}^{\tilde{a}\tilde{b}n} + \frac{3}{2}F^{\tilde{a}\tilde{b}}\right\}$;

2) $B_{\tilde{a}\tilde{b}}^{\tilde{c}} = 0$, $B_{\tilde{a}\tilde{b}\tilde{c}} = C_{\tilde{a}\tilde{b}\tilde{c}}$, $\sigma_{\tilde{a}}^{\tilde{b}} = -\sqrt{-1}\left\{\sqrt{2}\,B_{\tilde{a}n}^{\tilde{b}} + \delta_{\tilde{a}}^{\tilde{b}}\right\}$, $\sigma_{\tilde{a}\tilde{b}} = \sqrt{-1}\left\{\sqrt{2}\,\tilde{B}_{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}B_{\tilde{a}\tilde{b}}^n + \frac{1}{\sqrt{2}}\tilde{B}_{\tilde{a}\tilde{b}n} + \frac{3}{2}F_{\tilde{a}\tilde{b}}\right\}$;

3) $\sqrt{2}\,B_{n\tilde{a}\tilde{b}} = F_{\tilde{a}\tilde{b}}$, $\sqrt{2}\,B^{n\tilde{a}\tilde{b}} = F^{\tilde{a}\tilde{b}}$, $\sigma_{\tilde{b}}^{\tilde{a}} = \frac{-\sqrt{-1}}{\sqrt{2}}\left\{B_{\tilde{b}}^{n\tilde{a}} - B_{n\tilde{b}}^{\tilde{a}}\right\}$, $\sigma_{n\tilde{b}} = \sqrt{-1}\left\{\tilde{B}_{n\tilde{b}n} + B_{n\tilde{b}}^n\right\}$, $\sigma_n^{\tilde{b}} = -\sqrt{-1}\left\{\tilde{B}^{n\tilde{b}n} + B_n^{n\tilde{b}}\right\}$.

Since $B_{\tilde{c}}^{\tilde{a}\tilde{b}} = B_{\tilde{a}\tilde{b}}^{\tilde{c}} = 0$, then $\alpha^{\tilde{a}} = \alpha_{\tilde{a}} = 0$. So, we have $B_n^{\tilde{a}\tilde{b}} = B_{\tilde{a}\tilde{b}}^n = B_n^{n\tilde{b}} = B_{n\tilde{b}}^n = 0$. Now, since $\sigma$ is symmetric then $\sigma_{[n\tilde{b}]} = 0$. Regards item 3) above we get $\tilde{B}_{[n\tilde{b}]n} = \tilde{B}_{n\tilde{b}n} = 0 = \tilde{B}^{[n\tilde{b}]n} = \tilde{B}^{n\tilde{b}n}$ and this implies that $\sigma_{n\tilde{b}} = \sigma_n^{\tilde{b}} = 0$. As well as, $\sigma_{[\tilde{a}\tilde{b}]} = \sigma^{[\tilde{a}\tilde{b}]} = 0$ and this implies that $\sqrt{2}\tilde{B}^{n[\tilde{a}\tilde{b}]} + \frac{1}{\sqrt{2}}\tilde{B}^{[\tilde{a}\tilde{b}]n} + \frac{3}{2}F^{[\tilde{a}\tilde{b}]} = 0$. Then, we get:

$$\sqrt{2}\,B^{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}\tilde{B}^{\tilde{a}\tilde{b}n} + \frac{3}{2}F^{\tilde{a}\tilde{b}} = 0 \implies \frac{1}{\sqrt{2}}\tilde{B}^{\tilde{a}\tilde{b}n} + \frac{5}{2}F^{\tilde{a}\tilde{b}} = 0.$$

So, we obtain the results. Also, equating the values of $\sigma_{\tilde{b}}^{\tilde{a}}$ in items 1), 3) and uses the above results, we have

$$\frac{-\sqrt{-1}}{\sqrt{2}}\left\{B_{\tilde{b}}^{n\tilde{a}} - B_{n\tilde{b}}^{\tilde{a}}\right\} = \sqrt{-1}\left\{\sqrt{2}B_{\tilde{b}}^{\tilde{a}n} + \delta_{\tilde{b}}^{\tilde{a}}\right\}, \quad B_{\tilde{b}}^{n\tilde{a}} + B_{n\tilde{b}}^{\tilde{a}} = \sqrt{2}\,\delta_{\tilde{b}}^{\tilde{a}}.$$

Since $B_{\tilde{b}}^{n\tilde{a}} = \alpha^{[n}\delta_{\tilde{b}}^{\tilde{a}]} = \frac{1}{2}\alpha^n\delta_{\tilde{b}}^{\tilde{a}}$ and $B_{n\tilde{b}}^{\tilde{a}} = \frac{1}{2}\alpha_n\delta_{\tilde{b}}^{\tilde{a}}$, then $\alpha^n + \alpha_n = 2\sqrt{2}$. So, from the above discussion we attain the requirement. $\rhd$

**Corollary 1.** *The hypersurface of $VG$-manifold does not has conditions more that the conditions of above theorem as well as $B^{\tilde{a}\tilde{b}\tilde{c}} = B_{\tilde{a}\tilde{b}\tilde{c}} = 0$, if it has $SGK_1$-structure.*

◁ The proof is the same of proof Theorem 4 and the addition condition is attain from Definition 5. ▷

**Theorem 5.** *The generalized Kenmotsu hypersurface* $(M^{2n-1}, \Phi, \xi, \eta, g)$ *of a Vaisman–Gray manifold is totally umbilical if and only if, it has*

$$\tilde{B}^{n\tilde{a}\tilde{b}} = B^{n\tilde{a}\tilde{b}}, \quad \tilde{B}_{n\tilde{a}\tilde{b}} = B_{n\tilde{a}\tilde{b}}, \quad \sigma_{nn} = \lambda = \sqrt{-1}\left\{1 - \frac{1}{\sqrt{2}}\alpha^n\right\}.$$

◁ Suppose that $M$ is totally umbilical $GK$-hypersurface of $VG$-manifold, then according to Definition 6, we have $\sigma_{ij} = \lambda g_{ij}$. So, if we use the values of $\sigma$ in Theorem 4 and elements of $g$ in Remark 1, that given by:

$$(g_{ij}) = \left(\begin{array}{c|c|c} \mathrm{O} & \mathrm{O} & I_{n-1} \\ \hline 0 & g_{nn} = 1 & 0 \\ \hline I_{n-1} & \mathrm{O} & \mathrm{O} \end{array}\right), \quad i, j = 1, 2, \ldots, 2n-1,$$

and substitute them in the last equation, we obtain

$$\sigma_{nn} = \lambda g_{nn} = \lambda, \ \sigma_{\tilde{b}}^{\tilde{a}} = \lambda g_{\hat{a}\tilde{b}} = \lambda \delta_{\tilde{b}}^{\tilde{a}} \implies \lambda = \sqrt{-1}\left\{1 - \frac{1}{\sqrt{2}}\alpha^n\right\}, \ \sigma_{\tilde{a}\tilde{b}} = \lambda g_{\tilde{a}\tilde{b}} = 0$$

$$\implies \frac{3}{2}F_{\tilde{a}\tilde{b}} + \sqrt{2}\,\tilde{B}_{n\tilde{a}\tilde{b}} + \frac{1}{\sqrt{2}}\,\tilde{B}_{\tilde{a}\tilde{b}n} = 0 \implies \frac{3}{2}F_{\tilde{a}\tilde{b}} + \sqrt{2}\,\tilde{B}_{n\tilde{a}\tilde{b}} - \frac{5}{2}F_{\tilde{a}\tilde{b}} = 0 \implies F_{\tilde{a}\tilde{b}} = \sqrt{2}\,\tilde{B}_{n\tilde{a}\tilde{b}}.$$

Since $F_{[\tilde{a}\tilde{b}]} = F_{\tilde{a}\tilde{b}}$ and $\tilde{B}_{n[\tilde{a}\tilde{b}]} = B_{n\tilde{a}\tilde{b}}$, then we have the result. The converse also hold true. ▷

**Corollary 2.** *The $GK$-hypersurface of $VG$-manifold is totally geodesic if and only if, it is totally umbilical with* $\alpha^n = \alpha_n = \sqrt{2}$.

◁ The proof is directly form Theorem 5, Definition 7 and Definition 6. ▷

**Theorem 6.** *The generalized Kenmotsu hypersurface of a Vaisman–Gray manifold is minimal if and only if, it has* $\sigma(\xi, \xi) = 0$.

◁ According to Definition 8, we getting on the following:

$$g^{ps}\sigma_{ps} = g^{\tilde{a}\tilde{b}}\sigma_{\tilde{a}\tilde{b}} + g^{\hat{a}\hat{b}}\sigma_{\hat{a}\hat{b}} + g^{\hat{a}\tilde{b}}\sigma_{\hat{a}\tilde{b}} + g^{\tilde{a}\hat{b}}\sigma_{\tilde{a}\hat{b}} + g^{nn}\sigma_{nn} = 0$$

$$\iff g^{\hat{a}\tilde{b}}\sigma_{\hat{a}\tilde{b}} + g^{\tilde{a}\hat{b}}\sigma_{\tilde{a}\hat{b}} + \sigma_{nn} = 0 \iff g^{\hat{a}\tilde{b}}\sigma_{\hat{a}\tilde{b}} + \overline{g^{\hat{a}\tilde{b}}}\,\overline{\sigma_{\hat{a}\tilde{b}}} + \sigma_{nn} = 0.$$

Then from Theorem 4, the foregoing equation reduce to $g^{\hat{a}\tilde{b}}\sigma_{\hat{a}\tilde{b}} - g^{\hat{a}\tilde{b}}\sigma_{\hat{a}\tilde{b}} + \sigma_{nn} = 0 \iff \sigma_{nn} = 0$. So, we have $\sigma(\xi, \xi) = 0$. ▷

**Theorem 7.** *The hypersurface of a Vaisman–Gray manifold has $SGK_2$-structure if and only if, $\sigma$ has the following components:*

$$\sigma_{\tilde{a}\tilde{b}} = \sqrt{-1}\,\sqrt{2}\,\tilde{B}_{n\tilde{a}\tilde{b}}, \quad \sigma_{n\tilde{b}} = \sigma_n^{\tilde{b}} = 0, \quad \sigma_{\tilde{a}}^{\tilde{b}} = -\sqrt{-1}\left\{1 - \frac{1}{\sqrt{2}}\alpha_n\right\}\delta_{\tilde{a}}^{\tilde{b}},$$

$$\sigma^{\tilde{a}\tilde{b}} = -\sqrt{-1}\,\sqrt{2}\,\tilde{B}^{n\tilde{a}\tilde{b}}, \quad \sigma_{\tilde{b}}^{\tilde{a}} = \sqrt{-1}\left\{1 - \frac{1}{\sqrt{2}}\alpha^n\right\}\delta_{\tilde{b}}^{\tilde{a}},$$

*where*

$$\alpha^{\tilde{a}} = \alpha_{\tilde{a}} = 0, \quad \alpha^n + \alpha_n = 2\sqrt{2},$$

$$B^{n\tilde{a}\tilde{b}} = \tilde{B}^{\tilde{a}\tilde{b}n} = B_{n\tilde{a}\tilde{b}} = \tilde{B}_{\tilde{a}\tilde{b}n} = 0, \quad B^{\tilde{a}\tilde{b}\tilde{c}} = C^{\tilde{a}\tilde{b}\tilde{c}}, \quad B_{\tilde{a}\tilde{b}\tilde{c}} = C_{\tilde{a}\tilde{b}\tilde{c}}, \quad \tilde{B}_{n\tilde{b}n} = \tilde{B}^{n\tilde{b}n} = 0.$$

◁ Definition 5 and Theorem 1 given Cartan's structure equations (first group) of $SGK_2$-manifold as follows:

$$d\omega^a = \omega_b^a \wedge \omega^b + C^{abc}\,\omega_b \wedge \omega_c - \delta_b^a\,\omega^b \wedge \omega,$$
$$d\omega_a = -\omega_a^b \wedge \omega_b + C_{abc}\,\omega^b \wedge \omega^c - \delta_a^b\,\omega_b \wedge \omega, \tag{3}$$
$$d\omega = 0.$$

Now, comparing equation 3 with Theorem 3 and take into account the technique of the proof of Theorem 4, we have the results. ▷

**Theorem 8.** *The hypersurface of a Vaisman–Gray manifold admits Kenmotsu structure if and only if, $\sigma$ has the same components in Theorem 7 under the same condition with* $B^{\tilde{a}\tilde{b}\tilde{c}} = B_{\tilde{a}\tilde{b}\tilde{c}} = 0.$

◁ The claim of this theorem is attains from Theorem 7 by using the definition of Kenmotsu manifold that mentioned in Definition 5. ▷

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## ОБ ОБОБЩЕННЫХ МНОГООБРАЗИЯХ КЕНМОЦУ КАК ГИПЕРПОВЕРХНОСТЯХ МНОГООБРАЗИЙ ВАЙСМАНА — ГРЕЯ

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**Аннотация.** В данной статье установлено, что гиперповерхности многообразий Вайсмана — Грея имеют обобщенные структуры Кенмоцу при некоторых условиях для формы Ли, тензоров Кириченко и второй фундаментальной формы погружения гиперповерхности в многообразие класса Вайсмана — Грея. Более того, компоненты второй фундаментальной формы определяются, когда вышеупомянутые гиперповерхности имеют обобщенные структуры Кенмоцу или какой-либо ее специальный вид или структуры Кенмоцу, такие, что некоторые из этих компонентов обращаются в ноль. При этом, некоторые компоненты формы Ли и некоторые компоненты тензоров Кириченко в классе Вайсмана — Грея обращаются в ноль. Изучается также минимальность вполне омбилических вполне геодезических гиперповерхностей многообразий Вайсмана — Грея с обобщенными структурами Кенмоцу. Кроме того, показано, что гиперповерхность многообразия Вайсмана — Грея, имеющего обобщенную структуру Кенмоцу, является вполне геодезической тогда и только тогда, когда она вполне омбилична и некоторые компоненты формы Ли постоянны.

**Ключевые слова:** почти контактные метрические многообразия, почти многообразия Кенмоцу, многообразия Вайсмана — Грея, гиперповерхности почти эрмитовых многообразий.

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