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WELL-POSEDNESS OF THE NONLOCAL BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS#

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Dedicated to the 85th anniversary of E. M. Semenov

Abstract. In this paper, we investigate a class of elliptic nonlocal boundary value problems posed in a Banach space and subject to the Samarskii–Ionkin boundary condition, with the principal part of the equation generated by a positive linear operator. The analysis is carried out within an abstract operator-theoretic framework, which allows us to treat a wide range of concrete problems arising in applications in a unified manner. First, we prove the fundamental well-posedness of the corresponding differential problem by establishing the existence, uniqueness, and continuous dependence of solutions on the given data. Particular attention is paid to the role of the positivity of the operator in ensuring these properties. Furthermore, we obtain coercive stability estimates for the solutions, which provide quantitative bounds in appropriate normed spaces. These estimates are derived for four distinct classes of elliptic nonlocal boundary value problems with Samarskii–Ionkin conditions, thereby demonstrating the robustness and generality of the proposed approach.

Keywords: Samarskii–Ionkin condition, elliptic equation, well-posedness, positive operator.

AMS Subject Classification: 35L10, 35L90, 35B35.

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1. Introduction

Elliptic partial differential equations play a fundamental role in nearly all branches of mathematics, including harmonic analysis, geometry, and Lie theory, and they are also of great importance in many applications in physics and engineering. One of the central issues in this field is the well-posedness of local boundary value problems. A typical example is the elliptic equation

$$-v''(t) + Av(t) = f(t) \quad (0 \leqslant t \leqslant T), \quad v(0) = v_0, \ v(T) = v_T \tag{1}$$

posed in an arbitrary Banach space E, where A is a positive operator. This problem, together with its various applications, has been extensively studied in the literature (see, for example, [1–4], and the references therein).

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In mathematical modeling, elliptic equations are usually supplemented with local boundary conditions that prescribe the values of the solution on the boundary of the domain. However, in many practical situations, classical boundary conditions do not adequately describe the underlying physical, chemical, biological, or environmental processes. In such cases, nonlocal boundary conditions are more appropriate. These conditions typically arise when boundary data cannot be measured directly or when the boundary values depend on the behavior of the solution inside the domain. As a result, nonlocal boundary value problems have received considerable attention, and their well-posedness has been widely investigated for both partial differential and difference equations (see, e. g., [1–3], [5–30], and the references therein).

In the paper [31], Ionkin studied a nonlocal problem for a one-dimensional parabolic equation arising in modeling some nonclassical thermal processes. The existence of solutions was proved and, later, their stability was established in [32]. For a parabolic equation in one space variable, Samarskii [33] proposed a nonlocal boundary value problem formulation covering both classical initial-boundary value problems and Ionkin's problem from [31, 32].

Recently, various nonlocal boundary value problems with Samarskii–Ionkin type condition for partial differential equations have been investigated by many researchers (see, e. g. [34–39], and the references given therein).

In the paper [40], the nonlocal boundary value problem with Samarskii–Ionkin type condition for the elliptic equation

$$\begin{cases} -\dfrac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0 < t < T, \\ u(0) = \varphi,\ u'(0) = u'(T) + \mu u(T) + \psi, \end{cases} \tag{2}$$

in a Banach space E with the positive operator A and $\mu \geqslant 0$ was investigated.

A function $u(t)$ is said to be a solution of problem (2) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the interval $(0, T)$ and continuously differentiable on the closed interval $[0, T]$. The derivatives at the endpoints are understood in the sense of one-sided derivatives.

(ii) For every $t \in [0, T]$, the element $u(t)$ belongs to the domain $D(A)$, and the function $Au(t)$ is continuous on $[0, T]$.

(iii) The function $u(t)$ satisfies the differential equation and the nonlocal boundary condition (2).

In this paper, we investigate the well-posedness of the nonlocal boundary value problem (2). The main well-posedness theorems are established throughout the paper. In addition, we present applications and derive new coercive stability estimates for the solutions of four classes of elliptic nonlocal boundary value problems.

2. Auxiliary Results

In this section, we give some auxiliary statements from [1] which will be useful in the sequel.

Definition. The operator A is said to be positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle ϕ, $0 < 2\phi < \pi$, symmetric with respect to the real axis and if on the edges of this sector, $S_1(\phi) = \{\rho e^{i\phi} : 0 \leqslant \rho < \infty\}$ and $S_2(\phi) = \{\rho e^{-i\phi} : 0 \leqslant \rho < \infty\}$ and outside of it the resolvent $(\lambda I - A)^{-1}$ is subject to the bound

$$\left\|(\lambda I - A)^{-1}\right\|_{E \to E} \leqslant \frac{M(\phi)}{1 + |\lambda|}. \tag{3}$$

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A, E)$. The operator A is said to be strongly positive in a Banach space E, if $\varphi(A, E) < \pi/2$ (see [41]).

The operator $B = A^{1/2}$ has better spectral properties than the positive operator A. Indeed, the operator $(-B)$ is a generator of an analytic semigroup $\exp\{-tBt\}$ $(t \geqslant 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$, i. e. the following estimates

$$\left\| \exp(-tB) \right\|_{E\to E}, \left\| tB \exp(-tB) \right\|_{E\to E} \leqslant M(B) e^{-\delta(B)t} \quad (t > 0) \tag{4}$$

hold for some number $M(B) \in [1, +\infty)$, $\delta(B) \in (0, +\infty)$. From that it follows that the operator $I - e^{-2TB}$ has the bounded inverse and the following estimate holds:

$$\left\| \left(I - e^{-2TB}\right)^{-1} \right\|_{E\to E} \leqslant M(B) \left(1 - e^{-2T\delta(B)}\right)^{-1}. \tag{5}$$

The following formula

$$\begin{aligned} v(t) = R\Bigg\{ & \left(e^{-tB} - e^{-(2T-t)B}\right) v_0 + \left(e^{-(T-t)B} - e^{-(T+t)B}\right) v_T \\ & - \left(e^{-(T-t)B} - e^{-(T+t)B}\right) (2B)^{-1} \int_0^T \left(e^{-(T-s)B} - e^{-(T+s)B}\right) f(s)\, ds \Bigg\} \\ & + (2B)^{-1} \int_0^T \left(e^{-|t-s|B} - e^{-(t+s)B}\right) f(s)\, ds \end{aligned} \tag{6}$$

holds for the exact solution of the problem (1) under sufficiently smooth data v_0, v_T and $f(t)$. Here and in this paper we will put $R = (I - e^{-2TB})^{-1}$.

Let, we denote by $C_{0T}^{\beta,\gamma}(E)$ $(0 \leqslant \gamma \leqslant \beta,\ 0 < \beta < 1)$, the Banach space obtained by completion of the set of all smooth E-valued functions $\varphi(t)$ on $[0, T]$ with respect to the norm

$$\|\varphi\|_{C_{0T}^{\beta,\gamma}(E)} = \|\varphi\|_{C(E)} + \sup_{0\leqslant t<t+\tau\leqslant T} \frac{(t+\tau)^\gamma (T-t)^\gamma \left\|\varphi(t+\tau) - \varphi(t)\right\|_E}{\tau^\beta}.$$

Here $C(E)$ stands for the Banach space of all continuous E-valued functions $\varphi(t)$ on $[0, T]$ equipped with the norm

$$\|\varphi\|_{C(E)} = \max_{0\leqslant t\leqslant T} \|\varphi(t)\|_E.$$

To formulate the well-posedness of the problem (1), we introduce the Banach space $E_\alpha = E_\alpha(B, E)$ $(0 < \alpha < 1)$ consists of all $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \left\| B \exp\{-zB\} v \right\|_E + \|v\|_E$$

is finite.

Theorem 1 [1]. *Let A is the positive operator in a Banach space E and $v_0'', v_T'' \in E_{\beta-\gamma} \cup E_0^{\beta,\gamma}$, $f(t) \in C_{0T}^{\beta,\gamma}(E)$ $(0 \leqslant \gamma \leqslant \beta,\ 0 < \beta < 1)$. Then for the solution $v(t)$ in $C_{0T}^{\beta,\gamma}(E)$ of the boundary value problem* (1) *the coercive inequalities*

$$\left\| v'' \right\|_{C(E_{\beta-\gamma})} \leqslant M \left[\beta^{-1} \left[\left\| v_0'' \right\|_{E_{\beta-\gamma}} + \left\| v_T'' \right\|_{E_{\beta-\gamma}} \right] + \beta^{-1}(1-\beta)^{-1} \|f\|_{C_{0T}^{\beta,\gamma}(E)} \right], \tag{7}$$

$$\begin{aligned}&\left\|v''\right\|_{C^{\beta,\gamma}_{0T}(E)}+\left\|Av\right\|_{C^{\beta,\gamma}_{0T}(E)}\\&\leqslant M\left[\left|v_0''\right|^{\beta,\gamma}_0+\left|v_T''\right|^{\beta,\gamma}_0+(\beta-\gamma)^{-1}(1-\beta)^{-1}\|f\|_{C^{\beta,\gamma}_{0T}(E)}\right],\quad \gamma\neq\beta\end{aligned}\tag{8}$$

hold, where M does not depend on β,γ,v_0'', v_T'' and $f(t)$. Here, $|w|^{\beta,\gamma}_0$ denotes norm of the Banach space $E^{\beta,\gamma}_0$ consists of those $w\in E$ for which the norm

$$\left|w\right|^{\beta,\gamma}_0=\max_{0\leqslant z\leqslant T}\left\|e^{-zB}w\right\|_E+\sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}(T-z)^{\gamma}\left\|\left(e^{-(z+\tau)B}-e^{-zB}\right)w\right\|_E$$

is finite.

Note that the parameter γ can be chosen freely in $[0,\beta]$, which increases the number of function spaces in which boundary value problem (1) is well posed. In particular, it is important that the boundary value problem (1) is well posed in the Hölder space $C^{\beta}(E)=C^{\beta,0}_{0T}(E)$ without a weight ($\gamma=0$).

3. Main Theorem on Well-Posedness of Problem (2)

Let us give lemma that will be needed below.

Lemma 1 [40]**.** *Let A be positive operator in a Banach space E then the operator $(I-(B-\mu)(B+\mu)^{-1}e^{-TB})$ has an inverse $(I-(B-\mu)(B+\mu)^{-1}e^{-TB})^{-1}$ and the following estimates hold:*

$$\left\|B(B+\mu)^{-1}\right\|_{E\to E},\left\|(B-\mu)(B+\mu)^{-1}\right\|_{E\to E}\leqslant M_1(B),\tag{9}$$

$$\left\|\left(I-(B-\mu)(B+\mu)^{-1}e^{-TB}\right)^{-1}\right\|_{E\to E}\leqslant\frac{M_2(B)}{1-M_1(B)e^{-\delta(B)T}}\,.\tag{10}$$

Here $\delta(B)>\frac{\ln M_1(B)}{T}$.

We consider the nonlocal boundary value problem (2). Using formula (6) and $u(0)=\varphi$, we get

$$\begin{aligned}u(t)=&\left(I-e^{-2TB}\right)^{-1}\left\{\left(e^{-tB}-e^{-(2T-t)B}\right)\varphi+\left(e^{-(T-t)B}-e^{-(T+t)B}\right)u(T)\right.\\&\left.-\left(e^{-(T-t)B}-e^{-(T+t)B}\right)(2B)^{-1}\int_0^T\left(e^{-(T-s)B}-e^{-(T+s)B}\right)f(s)\,ds\right\}\\&+(2B)^{-1}\int_0^T\left(e^{-|t-s|B}-e^{-(t+s)B}\right)f(s)\,ds.\end{aligned}\tag{11}$$

Taking the derivative, we get

$$\begin{aligned}u'(t)=&\left(I-e^{-2TB}\right)^{-1}\left\{-B\left(e^{-tB}+e^{-(2T-t)B}\right)\varphi+B\left(e^{-(T-t)B}+e^{-(T+t)B}\right)u(T)\right.\\&\left.-\left(e^{-(T-t)B}+e^{-(T+t)B}\right)2^{-1}\int_0^T\left(e^{-(T-s)B}-e^{-(T+s)B}\right)f(s)\,ds\right\}\\&+2^{-1}\left(-\int_0^t\left(e^{-(t-s)B}f(s)\,ds+\int_t^T e^{-(s-t)B}f(s)\,ds\right)+2^{-1}\int_0^T e^{-(t+s)B}\right)f(s)\,ds.\end{aligned}\tag{12}$$

Applying the formula (12) and the nonlocal condition $u'(0) = u'(T) + \mu u(T) + \psi$, we get

$$\begin{aligned}
&\big(I - e^{-2TB}\big)^{-1} \Bigg\{ - B\big(I + e^{-2TB}\big)\varphi + 2Be^{-TB}u(T) \\
&- e^{-TB} \int_0^T \Big(e^{-(T-s)B} - e^{-(T+s)B}\Big) f(s)\, ds \Bigg\} + \int_0^T e^{-sB} f(s)\, ds \\
&= \Big(I - e^{-2TB}\Big)^{-1} \Bigg\{ - 2Be^{-TB}\varphi + B\Big(I + e^{-2TB}\Big)u(T) \\
&- \Big(I + e^{-2TB}\Big) 2^{-1} \int_0^T \Big(e^{-(T-s)B} - e^{-(T+s)B}\Big) f(s)\, ds \Bigg\} \\
&- 2^{-1} \int_0^T e^{-(T-s)B} f(s)\, ds + 2^{-1} \int_0^T e^{-(T+s)B} f(s)\, ds + \mu u(T) + \psi
\end{aligned}$$

or

$$\begin{aligned}
&- (B + \mu)\Big(I - (B - \mu)(B + \mu)^{-1} e^{-TB}\Big) u(T) \\
&= B\Big(I - e^{-TB}\Big)\varphi + \Big(I + e^{-TB}\Big)\psi - \int_0^T \Big(e^{-sB} + e^{-(T-s)B}\Big) f(s)\, ds.
\end{aligned}$$

From that and Lemma 1 it follows that

$$\begin{aligned}
u(T) = -B(B + \mu)^{-1} &\Big(I - (B - \mu)(B + \mu)^{-1} e^{-TB}\Big)^{-1} \Bigg\{ \Big(I - e^{-TB}\Big)\varphi \\
&+ B^{-1}\Big(I + e^{-TB}\Big)\psi - B^{-1} \int_0^T \Big(e^{-sB} + e^{-(T-s)B}\Big) f(s)\, ds \Bigg\}.
\end{aligned} \tag{13}$$

It is easy to show that $u(t)$ defined on $[0, T]$ by formulas (11), and (13) is a unique solution in $C(E)$ of the problem (2) if, for example, $\varphi \in D(A^2)$, $\psi \in D(A^{\frac{3}{2}})$ and $Af(t) \in C(E)$ or $f'(t) \in C(E)$. Sufficient conditions for the well-posedness of the nonlocal boundary value problem (2) can be established if one considers this problem in certain spaces of smooth E — valued functions defined on $[0, T]$.

Note that for the solution of problem (2) the coercivity inequality

$$\big\|u''\big\|_{C(E)} + \big\|Au\big\|_{C(E)} \leqslant M_C\big[\, \|f\|_{C(E)} + \|A\varphi\|_E + \|A\psi\|_E\big]$$

fails. Nevertheless, we have the following result.

Theorem 2. *Let A is the positive operator in a Banach space E and $A\psi + \mu f(T) \in E$, $A\varphi - f(0) \in E_{\beta-\gamma} \cup E_0^{\beta,\gamma}$, $f(t) \in C_{0T}^{\beta,\gamma}(E)$ $(0 \leqslant \gamma \leqslant \beta, 0 < \beta < 1)$. Then for the solution $u(t)$ in $C_{0T}^{\beta,\gamma}(E)$ of the boundary value problem* (2) *the coercive inequalities*

$$\big\|u''\big\|_{C(E_{\beta-\gamma})} \leqslant M\Big[\beta^{-1}\Big[\|A\varphi - f(0)\|_{E_{\beta-\gamma}} + \|A\psi + \mu f(T)\|_E\Big] + \beta^{-1}(1 - \beta)^{-1}\|f\|_{C_{0T}^{\beta,\gamma}(E)}\Big],$$

$$\|u''\|_{C^{\beta,\gamma}_{0T}(E)} + \|Au\|_{C^{\beta,\gamma}_{0T}(E)} \leqslant M\Big[|A\varphi - f(0)|^{\beta,\gamma}_0 + \|A\psi + \mu f(T)\|_E + (\beta-\gamma)^{-1}(1-\beta)^{-1}\|f\|_{C^{\beta,\gamma}_{0T}(E)}\Big], \quad \gamma \neq \beta,$$

hold, where M does not depend on β, γ, φ, ψ and $f(t)$.

$$\|u\|_{C^{2+\alpha}_{0,T}(E)} + \|Au\|_{C^{\alpha}_{0,T}(E)} \leqslant M\left[\|A\varphi\|_E + \|B\psi\|_E + \frac{1}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}_{0,T}(E)}\right] \tag{14}$$

holds, where M does not depend on α, φ, ψ and $f(t)$.

◁ By Theorem 1 we have the following estimates

$$\|u''\|_{C(E_{\beta-\gamma})} \leqslant M\Big[\beta^{-1}\left[\|Au(0) - f(0)\|_{E_{\beta-\gamma}} + \|Au(T) - f(T)\|_{E_{\beta-\gamma}}\right] + \beta^{-1}(1-\beta)^{-1}\|f\|_{C^{\beta,\gamma}_{0T}(E)}\Big], \tag{15}$$

$$\|u''\|_{C^{\beta,\gamma}_{0T}(E)} + \|Au\|_{C^{\beta,\gamma}_{0T}(E)} \leqslant M\Big[|Au(0) - f(0)|^{\beta,\gamma}_0 + |Au(T) - f(T)|^{\beta,\gamma}_0 + \beta^{-1}(1-\beta)^{-1}\|f\|_{C^{\beta,\gamma}_{0T}(E)}\Big] \tag{16}$$

for the solution of the problem (2). We have that

$$Au(0) - f(0) = A\varphi - f(0).$$

Therefore, to prove the theorem it suffices to establish the estimates for $\|Au(T) - f(T)\|_{E_{\beta-\gamma}}$ and $|Au(T) - f(T)|^{\beta,\gamma}_0$. Applying formula (13), we get

$$\begin{aligned} Au(T) - f(T) = &-\big(I - (B-\mu)(B+\mu)^{-1}e^{-TB}\big)^{-1}B(B+\mu)^{-1} \\ &\times\Big\{\big(I - e^{-TB}\big)(A\varphi - f(0)) + B^{-1}\big(I + e^{-TB}\big)\big(B^2\psi + \mu f(T)\big) \\ &- \int_0^T Be^{-(T-s)B}(f(s) - f(T))\,ds - \int_0^T Be^{-sB}(f(s) - f(0))\,ds\Big\}. \end{aligned} \tag{17}$$

Then using the triangle inequality, the estimates (4), (9), (10), and the definition of the spaces $C^{\beta,\gamma}_{0T}(E)$ and $E_{\beta-\gamma}$, we get

$$\begin{aligned} &\big\|\lambda^{1-(\beta-\gamma)}Be^{-\lambda B}(Au(T) - f(T))\big\|_E \\ &\leqslant \big\|(B-\mu)(B+\mu)^{-1}\big\|_{E\to E}\big\|(I - (B-\mu)(B-\mu)^{-1}e^{-TB})^{-1}\big\|_{E\to E} \\ &\times\Bigg\{\big\|I - e^{-TB}\big\|_{E\to E}\lambda^{1-(\beta-\gamma)}\big\|Be^{-\lambda B}(A\varphi - f(0))\big\|_E + \big\|I + e^{-TB}\big\|_{E\to E}\lambda^{1-(\beta-\gamma)} \\ &\times\big\|e^{-\lambda B}(A\psi + \mu f(T))\big\|_E + \lambda^{1-(\beta-\gamma)}\int_0^T\big\|B^2e^{-(\lambda+T-s)B}\big\|_{E\to E}\|f(s) - f(T)\|_E\,ds \\ &+ \lambda^{1-(\beta-\gamma)}\int_0^T\big\|B^2e^{-(\lambda+s)B}\big\|_{E\to E}\|f(s) - f(0)\|_E\,ds\Bigg\} \leqslant M_3\Big[\|A\varphi - f(0)\|_{E_{\beta-\gamma}} \\ &+ \|A\psi + \mu f(T)\|_E\Big] + M_4\lambda^{1-\beta}\left(\int_0^T\frac{(T-s)^{\beta-1}}{\lambda+T-s}\,ds + \int_0^T\frac{s^{\beta-1}}{\lambda+s}\,ds\right)\|f\|_{C^{\beta,\gamma}_{0T}(E)} \end{aligned}$$

for any $\lambda > 0$. Since

$$\int_0^T \frac{\lambda^{1-\beta} s^{\beta-1}}{(\lambda+s)^2}\, ds \leqslant \int_0^\infty \frac{p^{\beta-1}}{1+p}\, dp \leqslant \frac{1}{\beta(1-\beta)},$$

we have that

$$\left\|\lambda^{1-(\beta-\gamma)} B e^{-\lambda B}(Au(T)-f(T))\right\|_{E_{\beta-\gamma}}$$
$$\leqslant M_3\left[\,\|A\varphi - f(0)\|_{E_{\beta-\gamma}} + \|A\psi + \mu f(T)\|_E\,\right] + M_4 \frac{1}{\beta(1-\beta)}\,\|f\|_{C_{0T}^{\beta,\gamma}(E)}$$

for any $\lambda > 0$. Therefore

$$\|Au(T)-f(T)\|_{E_{\beta-\gamma}} \leqslant M_3\left[\,\|A\varphi - f(0)\|_{E_{\beta-\gamma}} + \|A\psi + \mu f(T)\|_E\,\right] + M_4 \frac{1}{\beta(1-\beta)}\,\|f\|_{C_{0T}^{\beta,\gamma}(E)}.$$

Now, we estimate $|Au(T)-f(T)|_0^{\beta,\gamma}$. Using formula (1) and the triangle inequality, the estimates (4), (9), (10), and the definition of the spaces $C_{0T}^{\beta,\gamma}(E)$ and $E_0^{\beta,\gamma}$, we get

$$\left|Au(T)-f(T)\right|_0^{\beta,\gamma} = \max_{0\leqslant z\leqslant T}\left\|e^{-zB}(Au(T)-f(T))\right\|_E$$
$$+ \sup_{0\leqslant z<z+\tau\leqslant T} \tau^{-\beta}(z+\tau)^{\gamma}(T-z)^{\gamma}\left\|\left(e^{-(z+\tau)B}-e^{-zB}\right)(Au(T)-f(T))\right\|_E$$
$$\leqslant \left\|(B-\mu)(B+\mu)^{-1}\right\|_{E\to E}\left\|\left(I-(B-\mu)(B-\mu)^{-1}e^{-TB}\right)^{-1}\right\|_{E\to E}\Bigg\{\left\|I-e^{-TB}\right\|_{E\to E}$$
$$\times\Big[\max_{0\leqslant z\leqslant T}\left\|e^{-zB}(A\varphi-f(0))\right\|_E + \sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}(T-z)^{\gamma}\left\|(e^{-(z+\tau)B}-e^{-zB})(A\varphi-f(0))\right\|_E\Big]$$
$$+\left\|I+e^{-TB}\right\|_{E\to E}\Big[\max_{0\leqslant z\leqslant T}\left\|e^{-zB}B^{-1}(A\psi+\mu f(T))\right\|_E + \sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}(T-z)^{\gamma}$$
$$\times\left\|\left(e^{-(z+\tau)B}-e^{-zB}\right)B^{-1}(A\psi+\mu f(T))\right\|_E\Big] + \max_{0\leqslant z\leqslant T}\int_0^T \left\|Be^{-(z+T-s)B}\right\|_{E\to E}\left\|f(s)-f(T)\right\|_E ds$$
$$+\sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}\int_0^T \left\|B(e^{-(z+\tau)B}-e^{-zB})e^{-(T-s)B}\right\|_{E\to E}\left\|f(s)-f(T)\right\|_E ds$$
$$+\max_{0\leqslant z\leqslant T}\int_0^T \left\|Be^{-(z+s)B}\right\|_{E\to E}\left\|f(s)-f(0)\right\|_E ds + \sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}\int_0^T \left\|B\big(e^{-(z+\tau)B}\right.$$
$$\left.\left.-e^{-zB}\big)e^{-sB}\right\|_{E\to E}\|f(s)-f(0)\|_E\, ds\right\} \leqslant M_3\Big[\left|A\varphi-f(0)\right|_0^{\beta,\gamma} + \|A\psi+\mu f(T)\|_E\Big]$$
$$\leqslant M_4\left(\int_0^T \frac{ds}{(T-s)^{1-\beta+\gamma}} + \int_0^T \frac{ds}{s^{1-\beta+\gamma}}\right)\|f\|_{C_{0T}^{\beta,\gamma}(E)} + M_5 \sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}\tau^{\beta}(z+\tau)^{-\beta}$$
$$\times\int_0^T \frac{ds}{(z+T-s)^{1-\beta+\gamma}}\,\|f\|_{C_{0T}^{\beta,\gamma}(E)} + \sup_{0\leqslant z<z+\tau\leqslant T}\tau^{-\beta}(z+\tau)^{\gamma}\tau^{\beta}(z+\tau)^{-\beta}\int_0^T \frac{ds}{(z+s)^{1-\beta+\gamma}}\|f\|_{C_{0T}^{\beta,\gamma}(E)}$$
$$\leqslant M_3\Big[\left|A\varphi-f(0)\right|_0^{\beta,\gamma} + \|A\psi+\mu f(T)\|_E\Big] + M_4\frac{1}{(\beta-\gamma)(1-\beta)}\,\|f\|_{C_{0T}^{\beta,\gamma}(E)}.$$

Therefore

$$\left|Au(T)-f(T)\right|_0^{\beta,\gamma}\leqslant M_3\Big[\left|A\varphi-f(0)\right|_0^{\beta,\gamma}+\left\|A\psi+\mu f(T)\right\|_E\Big]+M_4\,\frac{1}{(\beta-\gamma)(1-\beta)}\,\|f\|_{C_{0T}^{\beta,\gamma}(E)}.$$

Theorem 2 is proved. ▷

Note that the parameter γ can be chosen freely in $[0,\beta)$, which increases the number of function spaces in which nonlocal boundary value problem (2) is well posed. In particular, Theorem 2 implies the well-posedness of the boundary value problem (2) in the Hölder space $C^\beta(E)$ without a weight ($\gamma=0$).

Theorem 3 [1, 40]**.** *Suppose $A\psi+\mu f(T)\in E$, $A\varphi-f(0)\in E_\beta$, $f(t)\in C^\beta(E)$ $(0<\beta<1)$. Then the boundary value problem* (2) *is well-posed in Hölder space $C^\beta(E)$, if A is the positive operator in Banach space E. For the solution $u(t)$ in $C^\beta(E)$ of the boundary value problem* (2) *the coercive inequality*

$$\begin{aligned}&\left\|u''\right\|_{C^\beta(E)}+\left\|Au\right\|_{C^\beta(E)}+\left\|u''\right\|_{C(E_\beta)}\\&\leqslant\frac{M}{\beta(1-\beta)}\,\|f\|_{C^\beta(E)}+\frac{M}{\beta}\left[\|A\varphi-f(0)\|_{E_\beta}+\|A\psi+\mu f(T)\|_E\right]\end{aligned}$$

holds, where M does not depend on β, φ, ψ and $f(t)$.

4. Applications of Main Theorem 2

Finally, we consider the applications of Theorem 2 to the elliptic equations.

First, we consider the nonlocal boundary value problems for the two dimensional elliptic equation

$$\begin{cases}-\dfrac{\partial^2u}{\partial y^2}-(a(x)u_x)_x+\delta u=f(y,x), & 0<y<T,\ 0<x<l,\\ u(0,x)=\varphi(x),\,u_y(0,x)=u_y(T,x)+\mu u(T,x)+\psi(x), & 0\leqslant x\leqslant l,\\ u(y,0)=u(y,l),\ u_x(y,0)=u_x(y,l), & 0\leqslant y\leqslant T,\end{cases}\tag{18}$$

where $\delta>0$ is a sufficiently large number and $a(x)$, $\varphi(x)$, $\psi(x)$ and $f(y,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantee the problem (18) has a smooth solution $u(y,x)$. We will assume that $a(x)\geqslant a>0$ and $a(l)=a(0)$. Here, $\delta>0$ is a sufficiently large number.

We introduce the Banach spaces $C^\theta[0,l]$ $(0<\theta<1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\|\,\varphi\,\|_{C^\theta[0,l]}=\|\,\varphi\,\|_{C[0,l]}+\sup_{0\leqslant x<x+\tau\leqslant l}\frac{|\varphi(x+\tau)-\varphi(x)|}{\tau^\theta},$$

where $C[0,l]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,l]$ with the usual max norm. It is known that the differential expression

$$A^xv=-(a(x)v'(x))'+\delta v(x)\tag{19}$$

define a positive operator A^x acting in $C^\theta[0,l]$ with domain $C^{\theta+2}[0,l]$ and satisfying the conditions $v(0)=v(l)$, $v_x(0)=v_x(l)$. Therefore, we can replace boundary value problems (18)

by the abstract boundary value problem (2). Using the results of Theorem 2, we can obtain the following result.

Theorem 4. *Assume that*

$$-(a(x)\psi_x(x))_x + \delta\psi(x) + \mu f(T,x) \in C^{\theta}[0,l],$$

$$-(a(x)\varphi_x(x))_x + \delta\varphi(x) - f(0,x) \in C^{2(\beta-\gamma)+\theta}[0,l] \cup E_0^{\beta,\gamma}\big(C^{\theta}[0,l]\big),$$

$$\text{and} \quad f(y,x) \in C_{0T}^{\beta,\gamma}\big(C^{\theta}[0,l]\big) \quad (0 \leqslant \gamma \leqslant \beta,\ 0 < \beta < 1).$$

Then, for the solution of the boundary value problem (18) *the following coercive inequalities are valid:*

$$\|u_{yy}\|_{C(C^{2(\beta-\gamma)+\theta}[0,l])} \leqslant M(\beta,\delta,\mu)\Big[\|f\|_{C_{0T}^{\beta,\gamma}(C^{\theta}[0,l])} + \big\| - a(\cdot)\psi''(\cdot) + \delta\psi(\cdot) + \mu f(T,\cdot)\big\|_{C^{\theta}[0,l]} + \big\| - (a(\cdot)\varphi')'(\cdot) + \delta\varphi(\cdot) - f(0,\cdot)\big\|_{C^{2(\beta-\gamma)+\theta}[0,l]}\Big],$$

$$\|u_{yy}\|_{C_{0T}^{\beta,\gamma}(C^{\theta}[0,l])} + \|u\|_{C_{0T}^{\beta,\gamma}(C^{2+\theta}[0,l])} \leqslant M(\beta,\delta,\mu)\Big[\|f\|_{C_{0T}^{\beta,\gamma}(C^{\theta}[0,l])} + \big\| - a(\cdot)\psi''(\cdot) + \delta\psi(\cdot) + \mu f(T,\cdot)\big\|_{C^{\theta}[0,l]} + \big\| - (a(\cdot)\varphi')'(\cdot) + \delta\varphi(\cdot) - f(0,\cdot)\big\|_{E_0^{\beta,\gamma}(C^{\theta}[0,l])}\Big], \quad \gamma \neq \beta,$$

where $M(\beta,\delta,\mu)$ *does not depend on* $\varphi(x), \psi(x), \gamma,$ *and* $f(y,x)$.

Second, we consider the nonlocal boundary value problem for the two dimensional elliptic equation with involution in x

$$\begin{cases} -\dfrac{\partial^2 u(y,x)}{\partial y^2} - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u(y,x)}{\partial x}\right) + \delta u(y,x) \\ \qquad\qquad - \beta\left(\dfrac{\partial}{\partial x}\left(a(-x)\,\dfrac{\partial u(y,-x)}{\partial x}\right)\right) = f(y,x), & 0 < y < T,\ x \in (-l,l), \\ u(0,x) = \varphi(x),\ u_y(0,x) = u_y(T,x) + \mu u(T,x) + \psi(x), & x \in [-l,l], \\ u(y,-l) = u(y,l) = 0, & 0 \leqslant y \leqslant T, \end{cases} \tag{20}$$

where $\delta > 0$ is a sufficiently large number and $a(x), \varphi(x), \psi(x)$ and $f(y,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantee the problem (20) has a smooth solution $u(y,x)$. We will assume that $a \geqslant a(x) = a(-x) \geqslant \sigma > 0$, $\sigma - a|\beta| \geqslant 0$.

Theorem 5. *Assume that*

$$-(a(x)\psi_x(x))_x + \delta\psi(x) + \mu f(T,x) \in L_2[-l,l],$$

$$-(a(x)\varphi_x(x))_x + \delta\varphi(x) - f(0,x) = 0, \quad x \in [-l,l],$$

$$\text{and} \quad f(y,x) \in C_{0T}^{\beta,\gamma}(L_2[-l,l]) \quad (0 \leqslant \gamma \leqslant \beta,\ 0 < \beta < 1).$$

Then, for the solution of the boundary value problem ((20)) *the following coercive inequalities are valid:*

$$\|u_{yy}\|_{C_{0T}^{\beta,\gamma}(L_2[-l,l])} + \|u\|_{C_{0T}^{\beta,\gamma}(W_2[-l,l])} \leqslant M(\beta,\delta,\mu)\Big[\|f\|_{C_{0T}^{\beta,\gamma}(L_2[-l,l])} + \big\| - a(\cdot)\psi''(\cdot) + \delta\psi(\cdot) + \mu f(T,\cdot)\big\|_{L_2[-l,l]}\Big], \quad \gamma \neq \beta,$$

where $M(\beta,\delta,\mu)$ *does not depend on* $\varphi(x), \psi(x), \gamma,$ *and* $f(y,x)$.

The proof of Theorem 4 is based on the abstract Theorem 2, the self-adjointness and the positivity in $L_2[-l,l]$ of a differential operator A^x defined by the formula (see [4])

$$A^x v(x) = -(a(x)v_x(x))_x - \beta(a(-x)v_x(-x))_x + \delta v(x)$$

with the domain $D(A^x) = \{u \in W_2^2[-l,l] : u(-l) = u(l) = 0\}$.

Third, let Ω be the unit open cube in the $n-$dimensional Euclidean space $\mathbf{R}^n$ $(0 < x_k < 1,$ $1 \leqslant k \leqslant n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, we consider the mixed boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\dfrac{\partial^2 u(y,x)}{\partial y^2} - \sum\limits_{r=1}^{n} \alpha_r(x) \dfrac{\partial^2 u(y,x)}{\partial x_r^2} + \delta u(y,x) = f(y,x), \\ x = (x_1, \ldots, x_n) \in \Omega,\ 0 < y < T, \\ u(0,x) = \varphi(x),\ u_y(0,x) = u_y(T,x) + \mu u(T,x) + \psi(x),\ x \in \overline{\Omega}, \\ u(y,x) = 0,\ x \in S, \end{cases} \tag{21}$$

where $\delta > 0$ is a sufficiently large number and $a_r(x), \varphi(x), \psi(x)$ and $f(y,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantee the problem (21) has a smooth solution $u(y,x)$. We will assume that $a_r(x) \geqslant a_0 > 0$.

We introduce the Banach spaces $C_{01}^{\theta}(\overline{\Omega})$ $(\theta = (\theta_1, \ldots, \theta_n),\ 0 < x_k < 1,\ k = 1, \ldots, n)$ of all continuous functions satisfying a Hölder condition with the indicator $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_k \in (0,1)$, $1 \leqslant k \leqslant n$ and with weight $x_k^{\theta_k}(1 - x_k - h_k)^{\theta_k}$, $0 \leqslant x_k < x_k + h_k \leqslant 1$, $1 \leqslant k \leqslant n$ which equipped with the norm

$$\|f\|_{C_{01}^{\theta}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})}$$

$$+ \sup_{\substack{0 \leqslant x_k < x_k + h_k \leqslant 1, \\ 1 \leqslant k \leqslant n}} \left| f(x_1, \ldots, x_n) - f(x_1 + h_1, \ldots, x_n + h_n) \right| \prod_{k=1}^{n} h_k^{-\theta_k} x_k^{\theta_k} (1 - x_k - h_k)^{\theta_k},$$

where $C(\overline{\Omega})$ is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|.$$

It is known that the differential expression

$$A^x v = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(y,x)}{\partial x^2} + \delta v(y,x) \tag{22}$$

defines a positive operator A^x acting on $C_{01}^{\theta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\theta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S. Therefore, we can replace boundary value problem (21) by the abstract boundary value problem (2). Using the results of Theorem 2, we can obtain the next result.

Theorem 6. *Assume that*

$$-\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 \psi(x)}{\partial x^2} + \delta \psi(x) + \mu f(T,x) \in C_{01}^{\theta}(\overline{\Omega}),$$

$$-\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 \varphi(x)}{\partial x^2} + \delta \varphi(x) - f(0,x) = 0, \quad x \in \overline{\Omega},$$

$f(y,x) \in C_{0T}^{\beta,\gamma}(C_{01}^{\theta}(\overline{\Omega}))$ $(0 \leqslant \gamma \leqslant \beta,\ 0 < \beta < 1)$. *Then, for the solution of the boundary value problem* (21) *the following coercive inequalities are valid:*

$$\|u_{yy}\|_{C_{0T}^{\beta,\gamma}(C_{01}^{\theta}(\overline{\Omega}))} + \sum_{r=1}^{n} \left\| \frac{\partial^2 u}{\partial x_r^2} \right\|_{C_{0T}^{\beta,\gamma}(C_{01}^{\theta}(\overline{\Omega}))}$$
$$\leqslant M(\beta,\delta,\mu)\Big[\|f\|_{C_{0T}^{\beta,\gamma}(C_{01}^{\theta}(\overline{\Omega}))} + \big\| -a(\cdot)\psi''(\cdot) + \delta\psi(\cdot) + \mu f(T,\cdot)\big\|_{C_{01}^{\theta}(\overline{\Omega})}\Big], \quad \gamma \neq \beta,$$

where $M(\beta,\delta,\mu)$ *does not depend on* $\varphi(x), \psi(x), \gamma,$ *and* $f(y,x)$.

Fourth, we consider the nonlocal boundary value problem on the range $\{0 \leqslant y \leqslant T, x \in \mathbf{R}^n\}$ for $2m$-order multidimensional elliptic equations

$$\begin{cases} -\dfrac{\partial^2 u}{\partial y^2} + \sum\limits_{|r|=2m} a_r(x) \dfrac{\partial^{|\tau|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta u(y,x) = f(y,x), \\ 0 < y < T,\ x, r \in \mathbf{R}^n,\ |r| = r_1 + \dots + r_n, \\ u(0,x) = \varphi(x),\ u_y(0,x) = u_y(T,x) + \mu u(T,x) + \psi(x),\ x \in \mathbf{R}^n, \end{cases} \tag{23}$$

where $\delta > 0$ is a sufficiently large number and $a_r(x)$, $\varphi(x)$, $\psi(x)$ and $f(y,x)$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (23) has a smooth solution $u(y,x)$.

We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)(i\xi_1)^{r_1} \dots (i\xi_n)^{r_n}, \quad \xi = (\xi_1, \dots, \xi_n) \in R^n \tag{24}$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \tag{25}$$

acting on functions defined on the space $\mathbf{R}^n$, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \leqslant (-1)^m B^x(\xi) \leqslant M_2 |\xi|^{2m} < \infty \tag{26}$$

for $\xi \neq 0$. The problem (23) has a unique smooth solution. This allows us to reduce the boundary value problem (23) to the boundary value problem (2) in a Banach space $E = C^{\theta}(R^n)$ of all continuous bounded functions defined on $\mathbf{R}^n$ satisfying a Hölder condition with the indicator $\theta \in (0,1)$ with a strongly positive operator $A^x = B^x + \delta I$ defined by (25).

Theorem 7. *Assume that*

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|\tau|}\psi(x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta\psi(x) + \mu f(T,x) \in C^{\theta}(R^n),$$

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|\tau|}\varphi(x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta\varphi(x) - f(0,x) \in C^{2m(\beta-\gamma)+\theta}(R^n) \cup E_0^{\beta,\gamma}\big(C^{\theta}(R^n)\big),$$

and $f(y,x) \in C_{0T}^{\beta,\gamma}(C^{\theta}(R^n))$ $(0 \leqslant \gamma \leqslant \beta, 0 < \beta < 1)$. *Then, for the solution of the boundary value problem* (23) *the following coercive inequalities are valid:*

$$\|u_{yy}\|_{C(C^{2(\beta-\gamma)+\theta}(R^n))} \leqslant M(\beta,\delta,\mu)\Big[\|f\|_{C_{0T}^{\beta,\gamma}(C^{\theta}(R^n))} + \big\| -a(\cdot)\psi''(\cdot) + \delta\psi(\cdot) + \mu f(T,\cdot)\big\|_{C^{\theta}(R^n)}$$
$$+ \big\| -(a(\cdot)\varphi')'(\cdot) + \delta\varphi(\cdot) - f(0,\cdot)\big\|_{C^{2(\beta-\gamma)+\theta}(R^n)}\Big],$$

$$\|u_{yy}\|_{C_{0T}^{\beta,\gamma}(C^\theta(R^n))} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|}u}{\partial x_1^{r_1} \dots \|x_n^{r_n}} \right\|_{C_{0T}^{\beta,\gamma}(C^\theta(R^n))}$$
$$\leqslant M(\beta,\delta,\mu)\Big[\|f\|_{C_{0T}^{\beta,\gamma}(C^\theta(R^n))} + \| - a(\cdot)\psi''(\cdot) + \delta\psi(\cdot) + \mu f(T,\cdot)\|_{C^\theta(R^n)}$$
$$+ \| - (a(\cdot)\varphi')'(\cdot) + \delta\varphi(\cdot) - f(0,\cdot)\|_{E_0^{\beta,\gamma}(C^\theta(R^n))}\Big], \quad \gamma \neq \beta,$$

where $M(\beta,\delta,\mu)$ does not depend on $\varphi(x), \psi(x), \gamma$, and $f(y,x)$.

The proof of Theorem 7 is based on the abstract Theorem 2 and the positivity of the operator A^x in $C^\theta(R^n)$, the structure of the fractional spaces $E_\alpha((A^x)^{1/2}, C(R^n))$ (see, [1, 42]), and the coercivity inequality for an elliptic operator A^x in $C^\theta(R^n)$.

5. Conclusion and Future Plans

1. In the present paper, the nonlocal boundary value problem with Samarskii–Ionkin condition for elliptic equations in a Banach space with a positive operator is investigated. The main theorem on well-posedness of this problem is proved. In practice, the coercive stability estimates for solution of four types of nonlocal boundary value problems with Samarskii–Ionkin condition for elliptic differential equations are obtained.

2. Investigate the high order of accuracy for the numerical solution of the nonlocal boundary value problem for elliptic partial differential equations (see [1]).

3. Investigate the uniform two-step difference schemes and asymptotic formulas for the solution of the nonlocal boundary value perturbation problem

$$\begin{cases} -\varepsilon^2 \dfrac{d^2u(t)}{dt^2} + Au(t) = f(t), \ 0 < t < T, \\ u(0) = \varphi, \ u'(0) = u'(T) + \mu u(T) + \psi, \end{cases}$$

for a linear elliptic equation in a Banach space E with the positive operator A and with $\varepsilon \in (0,\infty)$ parameter multiplying the highest order derivative term (see [1, 43]).

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КОРРЕКТНОСТЬ НЕЛОКАЛЬНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ЭЛЛИПТИЧЕСКИХ УРАВНЕНИЙ

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Аннотация. В данной работе мы исследуем класс эллиптических нелокальных краевых задач в банаховом пространстве, где главной частью задачи является сильно положительный линейный оператор, порожденный граничным условием Самарского — Ионкина. Анализ проводится в рамках абстрактной операторно-теоретической модели, что позволяет рассматривать широкий круг конкретных задач, возникающих в приложениях. Сначала мы устанавливаем корректность соответствующей дифференциальной задачи, доказывая существование, единственность и непрерывную зависимость решений от начальных данных. Особое внимание уделяется роли сильной положительности оператора в обеспечении этих свойств. Кроме того, мы получаем оценки коэрцитивной устойчивости для решений, которые обеспечивают количественные оценки в соответствующих нормированных пространствах. Эти оценки исследуются для четырех различных классов эллиптических нелокальных краевых задач с условиями Самарского — Ионкина, демонстрируя тем самым устойчивость и общность предложенного подхода.

Ключевые слова: условие Самарского — Ионкина, эллиптическое уравнение, корректность постановки задачи, положительный оператор.

AMS Subject Classification: 35L10, 35L90, 35B35.

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