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RIEMANN SOLITONS ON LORENTZIAN PR-WAVE MANIFOLDS

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Abstract. In this research, we delve into the study of Riemann solitons within the context of Lorentzian pr-wave manifolds. Our investigation aims to systematically classify all vector fields that are associated with Riemann solitons on these specific manifolds. A significant aspect of our work is to identify which of these vector fields can be categorized as the gradient type. Furthermore, we demonstrate that any vector field that has the potential to be linked with Riemann solitons can be classified into several categories, namely, Ricci bi-conformal, Killing, and Ricci collineation vector fields. This classification not only enhances our understanding of the geometric properties of these manifolds but also sheds light on the intricate relationships between the various types of vector fields and Riemann solitons. The analysis also gives explicit conditions for the steady and non-steady cases. Gradient Riemann solitons occur only in the steady case, while the potential vector fields may additionally be Killing, Ricci collineation, or Ricci bi-conformal vector fields.

Keywords: Lorentzian pr-wave manifolds, Riemann soliton, Ricci bi-conformal vector.

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1. Introduction

Ricci flow, introduced by Hamilton [1] in the late 20th century, serves as a generalization of the Einstein metric. Specifically, for a pseudo-Riemannian manifold denoted as $(M, g(t))$, the Ricci flow is characterized by

$$\frac{\partial}{\partial t} g(t) = 2S_{g(t)},$$

where $S_{g(t)}$ represents the Ricci curvature tensor. The manifold (M, g) qualifies as a Ricci soliton [2] whenever

$$L_Y g + 2S = 2\lambda g,$$

for some vector field Y and a real constant λ . In cases where $Y = 0$, the Ricci solitons correspond to Einstein manifolds. If $Y = \text{grad } h$ for some smooth function h , then (M, g) is classified as a gradient Ricci soliton. Initially explored in the context of Riemannian geometry, Ricci solitons have also been examined under pseudo-Riemannian conditions, particularly focusing on the Lorentzian scenario [3, 4]. Various physical implications and applications of both the Ricci flow and Ricci solitons have been discussed in the literature [5–9].

The notion of Riemann flow on the manifold (M, g) , characterized by the Riemann curvature tensor R , was introduced by Udriște [10, 11] and is defined by the equation

$$\frac{\partial}{\partial t} \mathcal{G}(t) = -2R(g(t)),$$

where $\mathcal{G} = \frac{1}{2}g \odot g$ and for two $(0, 2)$ -tensors, ω and ϑ the symbol $\odot$ is defined by:

$$\begin{aligned} (\omega \odot \vartheta)(W_1, W_2, U_1, U_2) &= \omega(W_1, U_2)\vartheta(W_2, U_1) + \omega(W_2, U_1)\vartheta(W_1, U_2) \\ &\quad - \omega(W_1, U_1)\vartheta(W_2, U_2) - \omega(W_2, U_2)\vartheta(W_1, U_1), \end{aligned}$$

applicable to all vector fields W_1, W_2, U_1, U_2 . The manifold (M^n, g) is called a Riemann soliton [12] and is represented as (M^n, g, μ, V) whenever

$$2R + \mu g \odot g + g \odot \mathcal{L}_V g = 0, \quad (1.1)$$

for a specific vector field V and a constant μ . The classification of the Riemann soliton as expanding, shrinking, or steady is determined by the sign of μ , where $\mu > 0$ indicates expansion, $\mu < 0$ indicates contraction, and $\mu = 0$ signifies a steady state. If $V = \text{grad } h$ for some smooth function h , the soliton is termed a gradient Riemann soliton, which leads to the modified equation

$$2R + \mu g \odot g + 2g \odot \nabla^2 f = 0.$$

From (1.1), we have

$$\begin{aligned} 2R(W_1, W_2, U_1, U_2) &= -2\mu[g(W_1, U_2)g(W_2, U_1) - g(W_1, U_1)g(W_2, U_2)] \\ &\quad - [g(W_1, U_2)\mathcal{L}_V g(W_2, U_1) + g(W_2, U_1)\mathcal{L}_V g(W_1, U_2)] \\ &\quad + [g(W_1, U_1)\mathcal{L}_V g(W_2, U_2) + g(W_2, U_2)\mathcal{L}_V g(W_1, U_1)]. \end{aligned} \quad (1.2)$$

By contracting W_1 and U_2 in the previous equation, we obtain

$$2S(W_2, U_1) = -2((n-1)\mu + \text{div} V)g(W_2, U_1) - (n-2)\mathcal{L}_V g(W_2, U_1).$$

A variety of research has been undertaken on Riemann solitons in the framework of manifolds. For instance, Biswas et al. [13] explored Riemann solitons on almost co-Kähler manifolds, while Venkatesha et al. [14, 15] concentrated on Riemann solitons within the fields of contact geometry and nearly Kenmotsu manifolds. Additional information can be found in the studies cited in references [16–21]. Also, a vector field Y is referred to as a Ricci bi-conformal vector field, as described in [22], if it satisfies the conditions

$$\mathcal{L}_Y g = \alpha g + \beta S, \quad \mathcal{L}_Y S = \alpha S + \beta g,$$

for some smooth functions α and β . Also, see [23–27]. Ricci bi-conformal vector fields find applications across various fields, notably in General Relativity, where they elucidate the connection between gravitational fields and the geometric structure of spacetime. In Mathematical Physics, they are essential for analyzing solutions to Einstein's equations and their associated geometric attributes. Furthermore, in the field of Geometric Analysis, these vector fields contribute to the investigation of manifold properties by examining curvature and conformal structures.

On the other hand, a Lorentzian manifold that possesses a parallel light-like vector field is referred to as a Brinkmann wave, as noted in [28]. A Brinkmann wave manifold, denoted

as (M, g) , is classified as a pp-wave if $\text{tr}_{(3,5)(4,6)}(R \otimes R) = 0$, where R is the curvature tensor. In the work referenced as [29], Schimming demonstrated that a pp-wave manifold in $(n+2)$ dimensions can be expressed in coordinates $(x, y, z_1, \dots, z_n)$ where

$$g = 2dxdy + f(dy)^2 + \sum_{k=1}^n (dz_k)^2$$

with the condition that $\partial_x f = 0$. In the work referenced as [30], Leistner established an alternative characterization of pp-wave manifolds. Specifically, he demonstrated that a Brinkmann-wave manifold, which features a parallel light-like vector field X along with the induced parallel distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, qualifies as a pp-wave if and only if R fulfills the condition expressed in equation

$$R(U, V) : \mathcal{D}^\perp \rightarrow \mathcal{D} \quad (1.3)$$

for all vectors U, V in the tangent bundle TM . This condition can also be interpreted as $R(W_1, W_2) = 0$ for any vectors W_1, W_2 belonging to $\mathcal{D}^\perp$. Consequently, it can be inferred that pp-wave manifold exhibits Ricci isotropy, indicating that the image of the Ricci operator is entirely light-like and that the scalar curvature is zero, as noted in [30]. Additionally, Leistner proposed a novel category of non-irreducible Lorentzian manifolds that satisfy the condition in (1.3) but are associated with a recurrent vector field X , characterized by the relation $\nabla X = \omega \otimes X$, where ω is a one-form defined on M . These manifolds are referred to as pr-waves, as per [30]. Furthermore, a local coordinate description akin to that of pp-wave manifolds was provided in [30]: a Lorentzian manifold (M, g) of dimension $n+2 > 2$ is classified as a pr-wave if, in the vicinity of any point $o \in M$, there exist coordinates $(x, y, z_1, \dots, z_n)$ such that the metric g takes the form:

$$g = 2dxdy + f(dz)^2 + \sum_{k=1}^n (dz_k)^2, \quad (1.4)$$

where f is a smooth function on (M, g) . Batat in [31], examined the curvature characteristics and Ricci solitons associated with Lorentzian pr-wave manifolds.

The arrangement of this document is outlined as follows. In Section 2, we provide an in-depth examination of the geometric frameworks pertaining to Lorentzian pr-waves manifolds. Section 3 focuses on the identification of Riemann solitons found within these specific manifolds. Notably, we demonstrate that these manifolds are capable of hosting various vector fields, leading to the emergence of Riemann solitons, with a crucial finding that these Riemann solitons display gradient properties solely in the steady state.

2. PR-Waves Manifolds

Let ∇ denote the Levi-Civita connection of the manifold (M, g) . We denote its associated curvature tensor as R , adhering to the sign convention defined by the equation:

$$R(V, W) = \nabla_{[V, W]} - [\nabla_V, \nabla_W]$$

and the Ricci tensor S of g is described by $S(V, W) = \text{tr}(Z \rightarrow R(V, Z)W)$. In this paper, we will operate under the assumption that $n \geq 1$. Regarding the coordinate vector field basis $\{\partial_x, \partial_y, \partial_{z_1}, \dots, \partial_{z_n}\}$, for which equation (1.4) is applicable, the non-zero components of the metric are defined as

$$g_{xy} = 1, \quad g_{yy} = f, \quad g_{z_i z_i} = 1 \quad \text{for } 1 \leq i \leq n.$$

Therefore, the covariant derivatives of coordinate vector fields that may not vanish are expressed as

$$\begin{aligned}\nabla_{\partial_x}\partial_y &= \frac{1}{2}f_x\partial_x, \\ \nabla_{\partial_y}\partial_y &= \frac{1}{2}(ff_x + f_z)\partial_x - \frac{1}{2}f_x\partial_y - \frac{1}{2}\sum_{k=1}^n f_{z_k}\partial_{z_k}, \\ \nabla_{\partial_{z_i}}\partial_y &= \frac{1}{2}f_{z_i}\partial_x \quad \text{for } 1 \leq i \leq n.\end{aligned}\tag{2.1}$$

By inserting equation (2.1) into equation (1.3), we determine that the only curvature components that may be non-zero, for $1 \leq i, j \leq n$, are

$$\begin{aligned}R(\partial_x, \partial_y)\partial_x &= -\frac{1}{2}f_{xx}\partial_x, \\ R(\partial_x, \partial_y)\partial_{z_i} &= R(\partial_{z_i}, \partial_y)\partial_x = -\frac{1}{2}f_{xz_i}\partial_x, \\ R(\partial_x, \partial_y)\partial_y &= -\frac{1}{2}ff_{xx}\partial_x + \frac{1}{2}f_{xx}\partial_y + \frac{1}{2}\sum_{k=1}^n f_{xz_k}\partial_{z_k}, \\ R(\partial_{z_i}, \partial_y)\partial_{z_j} &= -\frac{1}{2}f_{z_iz_j}\partial_x, \\ R(\partial_{z_i}, \partial_y)\partial_y &= -\frac{1}{2}ff_{xz_i}\partial_x + \frac{1}{2}f_{xz_i}\partial_y + \frac{1}{2}\sum_{k=1}^n f_{z_iz_k}\partial_{z_k}.\end{aligned}$$

Hence

$$R_{xyxy} = -\frac{1}{2}f_{xx}, \quad R_{xyz_iz_i} = -\frac{1}{2}f_{xz_i}, \quad R_{z_iz_iz_j} = \frac{1}{2}f_{z_iz_j}.\tag{2.2}$$

By applying equation (2.2), we obtain

$$\begin{aligned}S(\partial_x, \partial_y) &= \frac{1}{2}f_{xx}, \quad S(\partial_y, \partial_y) = \frac{1}{2}\left(ff_{xx} - \sum_{k=1}^n f_{z_kz_k}\right), \\ S(\partial_{z_i}, \partial_y) &= \frac{1}{2}f_{xz_i} \quad \text{for } 1 \leq i \leq n,\end{aligned}\tag{2.3}$$

being zero otherwise. We examine a general vector field represented as $V = X\partial_x + Y\partial_y + \sum_{k=1}^n Z_k\partial_{z_k}$ on Lorentzian pr-waves manifolds. Here, the components X, Y, Z_i for $i = 1, \dots, n$ are smooth functions of the coordinates $(x, y, z_1, \dots, z_n)$. We derive the following results:

$$\begin{aligned}(\mathcal{L}_V g)(\partial_x, \partial_x) &= 2\partial_x Y, \\ (\mathcal{L}_V g)(\partial_x, \partial_y) &= \partial_x X + f\partial_x Y + \partial_y Y, \\ (\mathcal{L}_V g)(\partial_x, \partial_{z_i}) &= \partial_x Z_i + \partial_{z_i} Y, \\ (\mathcal{L}_V g)(\partial_y, \partial_y) &= Xf_x + \sum_{k=1}^n Z_k f_{z_k} + Yf_y + 2(\partial_y X + f\partial_y Y), \\ (\mathcal{L}_V g)(\partial_y, \partial_{z_i}) &= \partial_{z_i} X + f\partial_{z_i} Y + \partial_y Z_i, \\ (\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_j}) &= \partial_{z_i} Z_j + \partial_{z_j} Z_i.\end{aligned}\tag{2.4}$$

The Lie derivative of the Ricci tensor S in the direction of V is expressed as

$$\begin{aligned}
(\mathcal{L}_V S)(\partial_x, \partial_x) &= (\partial_x Y) f_{xx}, \\
(\mathcal{L}_V S)(\partial_x, \partial_y) &= \frac{1}{2} (\partial_x X) f_{xx} + \frac{1}{2} \left(f f_{xx} - \sum_{k=1}^n f_{z_k z_k} \right) \partial_x Y + \frac{1}{2} \sum_{k=1}^n (\partial_x Z_k) f_{xz_k} \\
&\quad + \frac{1}{2} (\partial_y Y) f_{xx} + \frac{1}{2} X f_{xxx} + \frac{1}{2} Y f_{xxy} + \frac{1}{2} \sum_{k=1}^n Z_k f_{xxz_k}, \\
(\mathcal{L}_V S)(\partial_x, \partial_{z_i}) &= \frac{1}{2} (\partial_x Y) f_{xz_i} - \frac{1}{2} (\partial_{z_i} Y) f_{xx}, \\
(\mathcal{L}_V S)(\partial_y, \partial_y) &= \frac{1}{2} V \left(f f_{xx} - \sum_{k=1}^n f_{z_k z_k} \right) + (\partial_y X) f_{xx} \\
&\quad + (\partial_y Y) \left(f f_{xx} - \sum_{k=1}^n f_{z_k z_k} \right) + \sum_{k=1}^n (\partial_y Z_k) f_{xz_k}, \\
(\mathcal{L}_V S)(\partial_y, \partial_{z_i}) &= \frac{1}{2} V (f_{yz_i}) + \frac{1}{2} (\partial_y Y) f_{xz_i} - \frac{1}{2} (\partial_{z_i} X) f_{xx} \\
&\quad - \frac{1}{2} (\partial_{z_i} Y) \left(f f_{xx} - \sum_{k=1}^n f_{z_k z_k} \right) - \frac{1}{2} \sum_{k=1}^n (\partial_{z_i} Z_k) f_{xz_k}, \\
(\mathcal{L}_V S)(\partial_{z_i}, \partial_{z_j}) &= \frac{1}{2} (\partial_{z_i} Y) f_{xz_j} + \frac{1}{2} (\partial_{z_j} Y) f_{xz_i}.
\end{aligned} \tag{2.5}$$

3. Riemann Solitons on Lorentzian PR-Waves Manifolds

In the following, we explore the presence of Riemann solitons within Lorentzian pr-waves manifolds. According to the findings presented in (1.2), Lorentzian pr-waves manifold qualifies as a Riemann soliton with a potential vector field $V = X\partial_x + Y\partial_y + \sum_{k=1}^n Z_k\partial_{z_k}$ if and only if

$$\begin{aligned}
2R_{xyxy} &= -2\mu g(\partial_x, \partial_y)^2 - 2g(\partial_x, \partial_y)(\mathcal{L}_V g)(\partial_x, \partial_y) + g(\partial_y, \partial_y)(\mathcal{L}_V g)(\partial_x, \partial_x), \\
2R_{xyz_iy} &= -g(\partial_x, \partial_y)(\mathcal{L}_V g)(\partial_y, \partial_{z_i}) + g(\partial_y, \partial_y)(\mathcal{L}_V g)(\partial_x, \partial_{z_i}), \\
2R_{z_iyyz_i} &= -2\mu g(\partial_y, \partial_y)g(\partial_{z_i}, \partial_{z_i}) - g(\partial_{z_i}, \partial_{z_i})(\mathcal{L}_V g)(\partial_y, \partial_y) - g(\partial_y, \partial_y)(\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_i}), \\
2R_{z_iyyz_j} &= -g(\partial_y, \partial_y)(\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_j}), \quad i \neq j, \\
2R_{xz_ixz_i} &= g(\partial_{z_i}, \partial_{z_i})(\mathcal{L}_V g)(\partial_x, \partial_x), \\
2R_{xz_iyz_i} &= 2\mu g(\partial_x, \partial_y)g(\partial_{z_i}, \partial_{z_i}) + g(\partial_x, \partial_y)(\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_i}) + g(\partial_{z_i}, \partial_{z_i})(\mathcal{L}_V g)(\partial_x, \partial_y), \\
2R_{xz_iyz_j} &= g(\partial_x, \partial_y)(\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_j}), \quad i \neq j, \\
2R_{xyz_ix} &= g(\partial_y, \partial_x)(\mathcal{L}_V g)(\partial_x, \partial_{z_i}).
\end{aligned}$$

By utilizing equations (1.4) and (2.2) in the preceding equations, we obtain

$$\begin{aligned}
-f_{xx} &= -2\mu - 2(\mathcal{L}_V g)(\partial_x, \partial_y) + f(\mathcal{L}_V g)(\partial_x, \partial_x), \\
-f_{xz_i} &= -(\mathcal{L}_V g)(\partial_y, \partial_{z_i}) + f(\mathcal{L}_V g)(\partial_x, \partial_{z_i}), \\
f_{z_iz_i} &= -2\mu f - (\mathcal{L}_V g)(\partial_y, \partial_y) - f(\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_i}), \\
f_{z_iz_j} &= -f(\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_j}), \quad i \neq j, \\
0 &= (\mathcal{L}_V g)(\partial_x, \partial_x), \\
0 &= 2\mu + (\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_i}) + (\mathcal{L}_V g)(\partial_x, \partial_y),
\end{aligned}$$

$$\begin{aligned} 0 &= (\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_j}), \quad i \neq j, \\ 0 &= (\mathcal{L}_V g)(\partial_x, \partial_{z_i}). \end{aligned}$$

From the fourth and seventh equations of the above system, it is obtained $f_{z_i z_j} = 0$ for $i \neq j$. We also have

$$\begin{aligned} (\mathcal{L}_V g)(\partial_x, \partial_x) &= (\mathcal{L}_V g)(\partial_x, \partial_{z_i}) = (\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_j}) = 0, \quad i \neq j, \\ (\mathcal{L}_V g)(\partial_x, \partial_y) &= \frac{1}{2} f_{xx} - \mu, \\ (\mathcal{L}_V g)(\partial_y, \partial_{z_i}) &= f_{xz_i}, \\ (\mathcal{L}_V g)(\partial_y, \partial_y) &= -\mu f - f_{z_i z_i} + \frac{f}{2} f_{xx}, \\ (\mathcal{L}_V g)(\partial_{z_i}, \partial_{z_i}) &= -\mu - \frac{1}{2} f_{xx}. \end{aligned} \tag{3.1}$$

By substituting (2.4) into the equations mentioned above, we derive

$$\partial_x Y = 0, \tag{3.2}$$

$$\partial_x Z_i + \partial_{z_i} Y = 0, \tag{3.3}$$

$$\partial_{z_i} Z_j + \partial_{z_j} Z_i = 0, \quad i \neq j, \tag{3.4}$$

$$\partial_x X + f \partial_x Y + \partial_y Y = \frac{1}{2} f_{xx} - \mu, \tag{3.5}$$

$$\partial_{z_i} X + f \partial_{z_i} Y + \partial_y Z_i = f_{xz_i}, \tag{3.6}$$

$$X f_x + \sum_{k=1}^n Z_k f_{z_k} + Y f_y + 2(\partial_y X + f \partial_y Y) = -\mu f - f_{z_i z_i} + \frac{f}{2} f_{xx} \quad (\forall i \in \{1, \dots, n\}), \tag{3.7}$$

$$2\partial_{z_i} Z_i = -\mu - \frac{1}{2} f_{xx} \quad (\forall i \in \{1, \dots, n\}). \tag{3.8}$$

Equation (3.2) yields

$$Y = Y(y, z_1, \dots, z_n).$$

Deriving the equation (3.3) with respect to z_j for all $i, j = 1, 2, \dots, n$, we get

$$\partial_{z_j} \partial_x Z_i + \partial_{z_j} \partial_{z_i} Y = 0.$$

From (3.4), and considering that $\partial_{z_j} \partial_x Z_i = -\partial_{z_j} \partial_{z_i} Y$, we derive

$$\partial_{z_j} \partial_x Z_i = \partial_{z_j} \partial_{z_i} Y = 0, \quad \text{for all } i, j = 1, 2, \dots, n \text{ such that } i \neq j.$$

Also, (3.8) and considering $\partial_{z_j} \partial_x Z_i = 0$, we deduce $f_{xxz_i} = 0$ for $i = 1, \dots, n$. We derive (3.4) with respect to z_k for $k \neq i, j$, we find

$$\partial_{z_k} \partial_{z_j} Z_i + \partial_{z_k} \partial_{z_i} Z_j = 0,$$

and so

$$\partial_{z_i} \partial_{z_j} Z_k + \partial_{z_i} \partial_k Z_j = 0, \quad \partial_{z_j} \partial_{z_k} Z_i + \partial_{z_j} \partial_{z_i} Z_k = 0,$$

which give

$$\partial_{z_j} \partial_{z_k} Z_i = 0, \quad \text{for all } i, j, k = 1, 2, \dots, n \text{ such that } i \neq j, k \neq i, j.$$

Then we can write

$$Y = \sum_{i=1}^n A_i(y, z_i)$$

and

$$Z_i = -(\partial_{z_i} A_i(y, z_i))x + \sum_{\substack{k=1, \\ k \neq i}}^n B_{ik}(z_k, y)$$

for some smooth functions A_i and B_{ik} such that $\partial_{z_j} B_{ij} = \partial_{z_i} B_{ji}$. Inserting the last equation in (3.8), it follows that

$$-2(\partial_{z_i} \partial_{z_i} A_i(y, z_i))x = -\mu - \frac{1}{2} f_{xx}, \quad i = 1, 2, \dots, n.$$

Since $f_{xxz_i} = 0$, we get $\partial_{z_i} \partial_{z_i} A_i(y, z_i) = A(y)$, $i = 1, 2, \dots, n$, for some function A . Therefore,

$$A_i(y, z_i) = \frac{1}{2} A(y) z_i^2 + A_{i1}(y) z_i + A_{i2}(y)$$

and

$$f_{xx} = -2\mu + 4A(y)x$$

for some smooth functions A_{i1} and A_{i2} . We also get

$$Y = \frac{1}{2} A(y) \sum_{i=1}^n z_i^2 + \sum_{i=1}^n A_{i1}(y) z_i + \tilde{A}(y), \quad (3.9)$$

and

$$Z_i = -(A(y) z_i + A_{i1}(y))x + \sum_{\substack{k=1, \\ k \neq i}}^n B_{ik}(z_k, y), \quad (3.10)$$

where $\tilde{A}(y) = \sum_{i=1}^n A_{i2}(y)$. Equation (3.5) gives

$$\partial_x X = -\frac{1}{2} A'(y) \sum_{i=1}^n z_i^2 - \sum_{i=1}^n A'_{i1}(y) z_i - \tilde{A}'(y) - 2\mu + A(y)x$$

and consequently we obtain

$$X = -2\mu x + \frac{1}{2} A(y) x^2 - \frac{1}{2} A'(y) x \sum_{i=1}^n z_i^2 - x \sum_{i=1}^n A'_{i1}(y) z_i - \tilde{A}'(y) x + \tilde{X}(y, z_1, \dots, z_n) \quad (3.11)$$

for some smooth function $\tilde{X}$. Applying (3.9), (3.10), and (3.11) to (3.6), we arrive at

$$\partial_{z_i} \tilde{X} + f(A(y) z_i + A_{i1}(y)) - 2(A'(y) z_i + A'_{i1}(y))x + \sum_{\substack{k=1, \\ k \neq i}}^n \partial_y B_{ik}(z_k, y) = f_{xz_i}. \quad (3.12)$$

Taking derivative (3.12) with respect to z_j and using $f_{z_i z_j} = 0$ for $i \neq j$, we infer

$$\partial_{z_j} \partial_{z_i} \tilde{X} + f_{z_j}(A(y) z_i + A_{i1}(y)) + \sum_{\substack{k=1, \\ k \neq i}}^n \partial_{z_j} \partial_y B_{ik}(z_k, y) = 0. \quad (3.13)$$

By differentiating equation (3.12) twice with respect to x , we have

$$(-2\mu + 4A(y)x)(A(y) z_i + A_{i1}(y)) = 0 \quad (\forall i \in \{1, \dots, n\}).$$

Then $-2\mu + 4A(y)x = 0$ or $A(y)z_i + A_{i1}(y) = 0$ for $i = 1, \dots, n$. If $A(y)z_i + A_{i1}(y) = 0$, then $A(y) = A_{i1}(y) = 0$ for $i = 1, \dots, n$. Equation (3.13) becomes $\partial_{z_j}\partial_{z_i}\tilde{X} + \partial_{z_j}\partial_y B_{ij}(z_j, y) = 0$. Similarly, $\partial_{z_i}\partial_{z_j}\tilde{X} + \partial_{z_i}\partial_y B_{ji}(z_i, y) = 0$. Since $\partial_{z_i}B_{ji}(z_i, y) + \partial_{z_j}B_{ij}(z_j, y) = 0$, we obtain $\partial_{z_i}\partial_{z_j}\tilde{X} = \partial_{z_i}\partial_y B_{ji}(z_i, y) = 0$ for all $i \neq j$. Therefore, we can write

$$B_{ij}(z_j, y) = B_{ij}(y), \quad \tilde{X} = \sum_{k=1}^n \bar{X}_i(y, z_i),$$

and $\partial_{z_i}\bar{X}_i(y, z_i) + B'_i(y) = f_{xz_i}$ for some smooth functions $\bar{X}_i$ where $B_i(y) = \sum_{k=1, k \neq i}^n \partial_y B_{ik}(y)$. Now we have $\partial_{z_i}\partial_{z_i}\bar{X}_i = f_{xz_i z_i}$. Using (3.7) we deduce that $f_{z_i z_i} = f_{z_j z_j}$ for $i, j = 1, 2, \dots, n$. Hence,

$$\partial_{z_i}\partial_{z_i}\bar{X}_i = D(y), \quad \bar{X}_i = \frac{1}{2}D(y)z_i^2 + D_{i1}(y)z_i + D_{i2}(y)$$

for some smooth functions $D(y)$, D_{i1} , and D_{i2} . Consequently, we find

$$f_{xz_i} = D(y)z_i + D_{i1}(y) + B'_i(y), \quad \tilde{X} = \frac{1}{2}D(y)\sum_{k=1}^n z_k^2 + \sum_{k=1}^n D_{k1}(y)z_k + \bar{D}(y),$$

where $\bar{D} = \sum_{k=1}^n D_{k2}(y)$. Consequently, for any $i \in \{1, \dots, n\}$, it follows that

$$Y = \tilde{A}(y), \quad X = -2\mu x + \frac{1}{2}D(y)\sum_{k=1}^n z_k^2 + \sum_{k=1}^n D_{k1}(y)z_k + \bar{D}(y), \quad Z_i = B_i(y), \quad f_{xx} = -2\mu.$$

Next, we derive (3.7) with respect to x , we infer

$$(-2\mu x + \tilde{X})(-2\mu) + \sum_{k=1}^n Z_k f_{xz_k} + \tilde{A}(y)f_{xy} + 2f_x \tilde{A}'(y) = -f_{xz_i z_i} \quad (3.14)$$

and by differentiating again with respect to x , we have

$$\mu^2 - \mu \tilde{A}'(y) = 0.$$

Then $\mu = 0$ or $\mu = \tilde{A}'(y)$. If $\mu = 0$, then $f_{xz_i} = D(y)z_i + D_{i1}(y) + B'_i(y)$, $f_{xx} = 0$,

$$Y = \tilde{A}(y), \quad X = \frac{1}{2}D(y)\sum_{k=1}^n z_k^2 + \sum_{k=1}^n D_{k1}(y)z_k + \bar{D}(y), \quad Z_i = B_i(y), \quad (3.15)$$

and

$$\tilde{X}f_x + \sum_{k=1}^n Z_k f_{z_k} + \tilde{A}(y)f_y + 2(\partial_y \tilde{X} + f \tilde{A}'(y)) = -f_{z_i z_i}. \quad (3.16)$$

Now, if $\mu \neq 0$ then $\mu = \tilde{A}'(y)$ and $\tilde{A}(y) = \mu y + a$ for some constant a . Then, we obtain $f_{xz_i} = D(y)z_i + D_{i1}(y) + B'_i(y)$, $f_{xx} = -2\mu$,

$$Y = \mu y + a, \quad X = -2\mu x + \frac{1}{2}D(y)\sum_{k=1}^n z_k^2 + \sum_{k=1}^n D_{k1}(y)z_k + \bar{D}(y), \quad Z_i = B_i(y) \quad (3.17)$$

and

$$(-2\mu x + \tilde{X})f_x + \sum_{k=1}^n Z_k f_{z_k} + (\mu y + a)f_y + 2(\partial_y X + f\mu) = -2\mu f - f_{z_i z_i}. \quad (3.18)$$

If we consider $A(y)z_i + A_{i1}(y) \neq 0$ and $-2\mu + 4A(y)x = 0$ then $\mu = 0$, $A(y) = 0$, and $A_{i1}(y) \neq 0$. In this case, $f_{xx} = 0$,

$$Y = \sum_{i=1}^n A_{i1}(y)z_i + \tilde{A}(y), \quad X = -x \sum_{i=1}^n A'_{i1}(y)z_i - \tilde{A}'(y)x + \tilde{X},$$

$$Z_i = -A_{i1}(y)x + \sum_{\substack{k=1 \\ k \neq i}}^n B_{ik}(z_k, y),$$
(3.19)

$$\partial_{z_i} \tilde{X} + f A_{i1}(y) - 2A'_{i1}(y)x + \sum_{\substack{k=1 \\ k \neq i}}^n \partial_y B_{ik}(z_k, y) = f_{xz_i},$$
(3.20)

and

$$Xf_x + \sum_{k=1}^n Z_k f_{z_k} + Yf_y + 2(\partial_y X + f\partial_y Y) = -f_{xz_i}.$$
(3.21)

Theorem 3.1. *The Lorentzian pr-waves manifold with metric (1.4) is a Riemann soliton (M, g, μ, V) of type*

- I) *steady and V admits (3.15)–(3.16);*
- II) *steady and V admits (3.19)–(3.21);*
- III) *non-steady and V admits (3.17)–(3.18).*

From (1.4), we can conclude that Riemann soliton on Lorentzian pr-waves manifolds is gradient soliton with $V = \nabla h = -f\partial_x h\partial_x + \partial_y h\partial_x + \partial_x h\partial_y + \sum_{k=1}^n \partial_{z_k} h\partial_{z_k}$ for some smooth function h , if and only if

$$\begin{cases} \partial_x h = Y, \\ \partial_y h = X + fY, \\ \partial_{z_i} h = Z_i, \quad i = 1, 2, \dots, n. \end{cases}$$
(3.22)

By taking the derivative from the first equation and the second equation (3.22) with respect to y and x , respectively, we conclude that $\partial_y Y = \partial_x X + f_x Y$. Similarly, we have $\partial_{z_i} Y = \partial_x Z_i$, $\partial_{z_i} X + f_{z_i} Y + f\partial_{z_i} Y = \partial_y Z_i$, and $\partial_{z_i} Z_j = \partial_{z_j} Z_i$ for $i \neq j$. In the case I) the last equations imply that

$$\tilde{A}'(y) = f_x \tilde{A}(y), \quad D(y)z_i + D_{i1}(y) + f_{z_i} \tilde{A}(y) = B'_i(y).$$

If $\tilde{A}(y) = 0$ then $Y = 0$, $D(y) = 0$, $D_{i1}(y) = B'_i(y)$, and $f_{xz_i} = 2B'_i(y)$. Taking the derivative of (3.16) with respect to z_j for $j \neq i$, we get

$$B_j(y)f_x + 2 \left(\tilde{D}(y) + \sum_{k=1}^n B_k z_k \right) B'_j(y) + B_j(y)f_{z_j z_j} + 2B'_j(y) = 0$$

and by differentiating the previous equation once more with respect to z_k for $k \neq j$, we obtain

$$B_j(y)B'_k(y) + B'_j(y)B_k(y) = 0.$$

Therefore, $B_j(y) = b_j$ for some constants b_j , $j = 1, 2, \dots, n$. Hence, in this case we have

$$h = \int \tilde{D}(y)dy + \sum_{k=1}^n b_k z_k y + \sum_{k=1}^n b_k z_k + c$$
(3.23)

for some constant c . If $\tilde{A}(y) \neq 0$ then $f_{xz_i} = 0$ and $D(y) = 0$, $D_{i1}(y) = -B'_i(y)$. In this case, we get

$$h = \tilde{A}(y)x + \int f\tilde{A}(y)dy + \int \tilde{D}dy + \sum_{k=1}^n \left(\int B_k(y)dy \right) z_k + \sum_{k=1}^n B_k(y)z_k + c_1 \quad (3.24)$$

for some constant c_1 . For case II), $\partial_y Y = \partial_x X + f_x Y$ yields $-3\mu = f_x(\mu y + a)$ and consequently $\mu = 0$ which is a contradiction. For case III) $\partial_{z_i} Y = \partial_x Z_i$ leads to $A_{i1}(y) = 0$, again a contradiction. Therefore we conclude:

Corollary 3.1. *The Lorentzian pr-waves manifold with metric g is a steady gradient Riemann soliton with potential function h that satisfies (3.23) or (3.24).*

REMARK 3.1. A vector field Y is classified as a conformal if it satisfies $\mathcal{L}_Y g = 2fg$, where f is a smooth function. In the special case where $f = 0$, the vector field Y is referred to as a Killing vector field. Conformal vector fields have been extensively analyzed in works such as [32, 33, 34]. From (3.1) we conclude a potential vector field of Riemann soliton on Lorentzian pr-waves manifold is a Killing if and only if $\mu = 0$, and $f_{xx} = f_{xz_i} = f_{z_i z_i} = 0$.

REMARK 3.2. A vector field Y is referred to as a Ricci collineation if it satisfies the condition $\mathcal{L}_Y S = 0$, where $\mathcal{L}_Y$ represents the Lie derivative and S denotes the Ricci tensor. Based on Theorem 3.1 and equation (2.5), any potential vector field associated with a Riemann soliton on a Lorentzian pr-waves manifold is a Ricci collineation vector field if

$$nV(f_{z_1 z_1}) + 2n\tilde{A}'(y)f_{z_1 z_1} + 2\sum_{k=1}^n B'_k(y)f_{xz_k} = 0, \quad V(f_{yz_i}) + \tilde{A}'(y)f_{xz_i} = 0.$$

REMARK 3.3. According to Theorem 3.1 and equation (2.5), any potential vector field V associated with a Riemann soliton on a pr-wave manifold qualifies as a Ricci bi-conformal vector field for $\alpha = \beta = 0$ such that

$$nV(f_{z_1 z_1}) + 2n\tilde{A}'(y)f_{z_1 z_1} + 2\sum_{k=1}^n B'_k(y)f_{xz_k} = 0, \quad V(f_{yz_i}) + \tilde{A}'(y)f_{xz_i} = 0.$$

In the work of Batat [31], it was established that a pr-wave manifold (M, g) is

(i) conformally flat if and only if its defining function f can be represented as

$$f = F(y)x + H(y) + \sum_{k=1}^n \left(\frac{1}{2}G(y)z_k + L_k(y) \right) z_k;$$

(ii) flat if and only if

$$f = F(y)x + H(y) + \sum_{k=1}^n L_k(y)z_k,$$

where F, G, L_k are smooth functions.

Corollary 3.2. *Any Riemann soliton on conformally flat Lorentzian pr-waves manifolds is steady.*

References

1. Hamilton, R. S. The Ricci Flow on Surfaces, *Contemporary Mathematics*, vol. 71, Providence, RI, American Mathematical Society, 1988, pp. 237–262. DOI: 10.1090/conm/071/954419.
2. Calin, C. and Crasmareanu, M. From the Eisenhart Problem to Ricci Solitons in f -Kenmotsu Manifolds, *Bulletin of the Malaysian Mathematical Sciences Society*, 2010, vol. 33, no. 3, pp. 361–368.

3. Brozos-Vázquez, M., Calvaruso, G., García-Río, E. and Gavino-Fernández, S. Three-Dimensional Lorentzian Homogeneous Ricci Solitons, *Israel Journal of Mathematics*, 2012, vol. 188, no. 1, pp. 385–403. DOI: 10.1007/s11856-011-0124-3.
4. Calvaruso, G. Homogeneous Structures on Three-Dimensional Lorentzian Manifolds, *Journal of Geometry and Physics*, 2007, vol. 57, no. 4, pp. 1279–1291. DOI: 10.1016/j.geomphys.2006.10.005.
5. Akbar, M. M. and Woolgar, E. Ricci Solitons and Einstein-Scalar Field Theory, *Classical and Quantum Gravity*, 2009, vol. 26, no. 5, article no. 055015, 14 pp. DOI: 10.1088/0264-9381/26/5/055015.
6. Chave, T. and Valent, G. Quasi-Einstein Metrics and Their Renormalizability Properties, *Helvetica Physica Acta*, 1996, vol. 69, no. 3, pp. 344–347. DOI: 10.5169/seals-116953.
7. Chave, T. and Valent, G. On a Class of Compact and Non-Compact Quasi-Einstein Metrics and Their Renormalizability Properties, *Nuclear Physics B*, 1996, vol. 478, no. 3, pp. 758–778. DOI: 10.1016/0550-3213(96)00341-0.
8. Friedan, D. H. Nonlinear Models in $2 + \epsilon$ Dimensions, *Annals of Physics*, 1985, vol. 163, no. 2, pp. 318–419. DOI: 10.1016/0003-4916(85)90384-7.
9. Kholodenko, A. L. Towards Physically Motivated Proofs of the Poincaré and Geometrization Conjectures, *Journal of Geometry and Physics*, 2008, vol. 58, no. 2, pp. 259–290. DOI: 10.1016/j.geomphys.2007.11.003.
10. Udriște, C. Riemann Flow and Riemann Wave, *Annals of West University of Timișoara. Series Mathematics and Informatics*, 2010, vol. 48, no. 1–2, pp. 265–274.
11. Udriște, C. Riemann Flow and Riemann Wave via Bialternate Product Riemannian Metric, arXiv:1112.4279 [math.AP], 2012. DOI: 10.48550/arXiv.1112.4279.
12. Hiriță, I. E. and Udriște, C. Ricci and Riemann Solitons, *Balkan Journal of Geometry and Its Applications*, 2016, vol. 21, no. 2, pp. 35–44.
13. Biswas, G. G., Chen, X. and De, U. C. Riemann Solitons on Almost Co-Kähler Manifolds, *Filomat*, 2022, vol. 36, no. 4, pp. 1403–1413. DOI: 10.2298/FIL2204403B.
14. Devaraja, M. N., Aruna Kumara, H. and Venkatesha, V. Riemann Soliton Within the Framework of Contact Geometry, *Quaestiones Mathematicae*, 2021, vol. 44, no. 5, pp. 637–651. DOI: 10.2989/16073606.2020.1732495.
15. Venkatesha, V., Aruna Kumara, H. and Naik, D. M. Riemann Solitons and Almost Riemann Solitons on Almost Kenmotsu Manifolds, *International Journal of Geometric Methods in Modern Physics*, 2020, vol. 17, no. 7, article no. 2050105. DOI: 10.1142/S0219887820501054.
16. Azami, S. and Jafari, M. Riemann Solitons on Relativistic Space-Times, *Gravitation and Cosmology*, 2024, vol. 30, no. 3, pp. 306–311. DOI: 10.1134/S020228932470021X.
17. Azami, S. and Jafari, M. Riemann Solitons on Perfect Fluid Spacetimes in $f(r, T)$ -Gravity, *Rendiconti del Circolo Matematico di Palermo. Series 2*, 2025, vol. 74, no. 1, article no. 2, 13 pp. DOI: 10.1007/s12215-024-01116-1.
18. Bakshi, M. R. and Baishya, K. K. Four Classes of Riemann Solitons on α -Cosymplectic Manifolds, *Afrika Matematika*, 2021, vol. 32, no. 3–4, pp. 577–588. DOI: 10.1007/s13370-020-00846-6.
19. Blaga, A. M. Remarks on Almost Riemann Solitons with Gradient or Torsion-Forming Vector Field, *Bulletin of the Malaysian Mathematical Sciences Society*, 2021, vol. 44, no. 5, pp. 3215–3227. DOI: 10.1007/s40840-021-01108-9.
20. Borges, V. On Complete Gradient Schouten Solitons, *Nonlinear Analysis*, 2022, vol. 221, article no. 112883, 15 pp. DOI: 10.1016/j.na.2022.112883.
21. De, K. and De, U. C. A Note on Almost Riemann Solitons and Gradient Almost Riemann Solitons, *Afrika Matematika*, 2022, vol. 33, no. 3, article no. 74, 10 pp. DOI: 10.1007/s13370-022-01010-y.
22. De, U. C., Sardar, A. and Sarkar, A. Some Conformal Vector Fields and Conformal Ricci Solitons on $N(k)$ -Contact Metric Manifolds, *AUT Journal of Mathematics and Computing*, 2021, vol. 2, no. 1, pp. 61–71. DOI: 10.22060/ajmc.2021.19220.1043.
23. Azami, S. and De, U. C. Ricci Bi-Conformal Vector Fields on Lorentzian Four-Dimensional Generalized Symmetric Spaces, *Filomat*, 2024, vol. 38, no. 9, pp. 3157–3164. DOI: 10.2298/FIL2409157A.
24. Azami, S. and De, U. C. Ricci Bi-Conformal Vector Fields on Lorentzian Five-Dimensional Two-Step Nilpotent Lie Groups, *Hacettepe Journal of Mathematics and Statistics*, 2024, vol. 53, no. 4, pp. 1118–1129. DOI: 10.15672/hujms.1294973.
25. Azami, S. and Fasihi-Ramandi, G. Ricci Bi-Conformal Vector Fields on Siklos Spacetimes, *Mathematics Interdisciplinary Research*, 2024, vol. 9, no. 1, pp. 45–76. DOI: 10.22052/MIR.2023.252926.1408.
26. Azami, S. and Jafari, M. Ricci Bi-Conformal Vector Fields on Homogeneous Gödel-Type Spacetimes, *Journal of Nonlinear Mathematical Physics*, 2023, vol. 30, no. 4, pp. 1700–1718. DOI: 10.1007/s44198-023-00151-3.

27. Sohrabpour, M. and Azami, S. Ricci Bi-Conformal Vector Fields on Lorentzian Walker Manifolds of Low Dimension, *Lobachevskii Journal of Mathematics*, 2023, vol. 44, no. 12, pp. 5437–5443. DOI: 10.1134/S1995080223120338.
28. Brinkmann, H. W. Einstein Spaces Which Are Mapped Conformally on Each Other, *Mathematische Annalen*, 1925, vol. 94, no. 1, pp. 119–145. DOI: 10.1007/BF01208647.
29. Schimming, R. Riemannsche Raume mit ebenfrontiger und mit ebener Symmetrie, *Mathematische Nachrichten*, 1974, vol. 59, no. 1-6, pp. 129–162. DOI: 10.1002/mana.19740590111.
30. Leistner, T. Conformal Holonomy of C -Spaces, Ricci-Flat, and Lorentzian Manifolds, *Differential Geometry and Its Applications*, 2006, vol. 24, no. 5, pp. 458–478. DOI: 10.1016/j.difgeo.2006.04.008.
31. Batat, W. Curvature Properties and Ricci Solitons of Lorentzian pr-Waves Manifolds, *Journal of Geometry and Physics*, 2014, vol. 75, pp. 7–16. DOI: 10.1016/j.geomphys.2013.08.014.
32. Carroll, S. M. *Spacetime and Geometry: An Introduction to General Relativity*, San Francisco, Addison Wesley, 2004, xiv+513 p.
33. Deshmukh, S. Geometry of Conformal Vector Fields, *Arab Journal of Mathematical Sciences*, 2017, vol. 23, no. 1, pp. 44–73. DOI: 10.1016/j.ajmsc.2016.09.003.
34. Yano, K. *The Theory of Lie Derivatives and Its Applications*, Mineola, NY, Dover Publications, 2020, x+299 p.

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РИМАНОВЫ СОЛИТОНЫ НА ЛОРЕНЦЕВЫХ pr-ВОЛНОВЫХ МНОГООБРАЗИЯХ

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Аннотация. В данном исследовании мы изучаем римановы солитоны в контексте лоренцевых многообразий типа PR-волн. Наша работа направлена на систематическую классификацию всех векторных полей, ассоциированных с римановыми солитонами на указанных многообразиях. Важным аспектом нашего исследования является выявление тех из них, которые могут быть отнесены к градиентному типу. Кроме того, мы показываем, что любое векторное поле, потенциально связанное с римановыми солитонами, может быть классифицировано по нескольким категориям: биконформное Риччи, киллингово и коллинеационное Риччи. Данная классификация не только углубляет наше понимание геометрических свойств этих многообразий, но и проясняет сложные взаимосвязи между различными типами таких полей и римановыми солитонами. Анализ также дает явные условия для стационарного и нестационарного случаев. Градиентные солитоны Римана возникают только в стационарном случае, в то время как потенциальные векторные поля могут дополнительно быть киллинговскими, коллинеационными Риччи или биконформными Риччи.

Ключевые слова: лоренцевы многообразия PR-волн, риманов солитон, биконформное Риччи векторное поле.

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