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BLOW-UP OF THE SOLUTION TO THE CAUCHY PROBLEM
FOR THE EQUATION OF SMALL OSCILLATIONS
OF AN ABSOLUTELY FLEXIBLE HOMOGENEOUS ROD

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Abstract. The Cauchy problem is studied for a nonlinear partial differential equation of Sobolev type (i. e., an equation not solved for the highest time derivative) that generalizes the equation of small oscillations of an absolutely flexible homogeneous rod. The problem is considered in the space of continuous functions defined on the entire real axis, for which limits exist at infinity. An explicit form of the classical solution to the corresponding linear homogeneous equation is obtained, and norm estimates for the operator-valued functions representing this solution are derived. An estimate for the norm of the solution to the Cauchy problem for the linear homogeneous equation is also given. The time interval of existence and uniqueness of a classical solution to the auxiliary Cauchy problem related to the original one is established, along with an estimate for the norm of this local solution. Conditions are found that ensure a correspondence between the classical solutions of the original and auxiliary Cauchy problems on a certain time interval. Moreover, conditions are examined under which the classical solution to the Cauchy problem blows up on a finite time interval.

Keywords: small oscillations of an absolutely flexible homogeneous rod, nonlinear Sobolev-type equation, blow-up of solution.

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1. Introduction

In this paper, we consider the Cauchy problem for the nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{a} \frac{\partial^4 u}{\partial t^2 \partial x^2} + b \left(\frac{1}{a} \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^2 u}{\partial x^2} + au \right) = \frac{\partial^2 u}{\partial x^2} f' \left(\frac{\partial u}{\partial x} \right), \quad (1)$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$. The prime in the equation denotes differentiation with respect to $u_x = \partial_x u = \partial u / \partial x$. The coefficients a and b are positive constants. The goal of this work is to derive conditions for the existence of a local classical solution to the Cauchy problem, and conditions under which the solution blows up within a finite time interval.

Equations of the form (1) belong to the class of pseudohyperbolic equations, as defined in monograph [1, Chapter 2]. In the special case where $(f(\cdot) \equiv 0)$, equation (1) describes small oscillations of an absolutely flexible homogeneous rod [2, Chapter 8, §43, (8.109)]. It is worth noting that such equations are often referred to as Sobolev-type equations in the literature, since the pioneering work of academician S. L. Sobolev initiated the deep study of partial differential equations that are not resolved with respect to the highest time derivative. Currently, a large number of studies are devoted to various problems involving such equations [3].

The Cauchy problem for equation (1) is considered in the space $C[\mathbb{R}]$ [4, Chapter VIII, §1] of continuous functions $g = g(x)$ for which both limits exist as $x \rightarrow \pm\infty$, and which are equipped with the norm $\|g\|_C = \sup_{x \in \mathbb{R}} |g(x)|$. The initial data are given by

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in \mathbb{R}, \quad (2)$$

and the sought classical solution $u = u(t, x)$, $(t, x) \in \overline{\mathbb{R}}_+ \times \mathbb{R}$, $\overline{\mathbb{R}}_+ = [0, +\infty)$, along with all partial derivatives appearing in equation (1), must for every t belong to $C[\mathbb{R}]$ with respect to x .

We denote by $C^{(k)}[\mathbb{R}] = \{g(x) \in C[\mathbb{R}] : g'(x), \dots, g^{(k)}(x) \in C[\mathbb{R}]\}$, for $k = 1, 2, \dots$, the subsets of $C[\mathbb{R}]$ consisting of k -times differentiable functions.

In equation (1), the nonlinearity $f(x)$, $x \in \mathbb{R}$, is assumed to be twice continuously differentiable and to satisfy the estimates

$$\sup_{x \in \mathbb{R}} |f^{(i)}(g(x))| \leq \left| f^{(i)} \left(\sup_{x \in \mathbb{R}} |g(x)| \right) \right|, \quad i = 0, 1 \quad (\forall g(x) \in C[\mathbb{R}]), \quad (3)$$

where $|f^{(0)}(r)| = |f(r)|$ and $|f^{(1)}(r)| = |f'(r)|$ for $r \in \overline{\mathbb{R}}_+$ are assumed to be non-decreasing functions.

We assume the rod is infinite. This idealization is acceptable [5] if optimal damping devices are placed at the ends of the rod, i.e., the boundary constraints are such that incident perturbations are not reflected.

The investigation of the Cauchy problem (1), (2) will be carried out according to the following plan:

1) We verify that the formulation of the Cauchy problem (1), (2) is correct and that a local-in-time classical solution exists. For this purpose, we first solve the corresponding linear homogeneous equation of type (1). We note that the Cauchy problem for linear Sobolev-type equations with constant coefficients is studied in detail in the fundamental monograph [1], where energy estimates and solvability theorems are obtained in various weighted Sobolev spaces. However, since we are dealing with the nonlinear equation (1), we employ a different approach to derive the solution of the corresponding linear equation and obtain the necessary estimates of auxiliary operator-valued functions. It is also worth mentioning, as pointed out in [6], that adding (or omitting) lower-order terms in an equation not resolved with respect to the highest derivative can significantly affect the solvability of the Cauchy problem.

2) We introduce the auxiliary Cauchy problem

$$v_{tt} - a^{-1}v_{ttxx} + b(a^{-1}v_{xxxx} + 2v_{xx} + av) = \partial_x^2 f(v), \quad (4)$$

with initial conditions

$$v|_{t=0} = \varphi'(x), \quad v_t|_{t=0} = \psi'(x), \quad x \in \mathbb{R}. \quad (5)$$

For the Cauchy problem (4), (5), we find a time interval $[0, t_1]$ during which a unique classical solution exists, and we estimate its norm in $C[\mathbb{R}]$.

3) We establish the connection between the classical solutions $u = u(t, x)$ and $v = v(t, x)$ of equations (1) and (4), respectively, on the interval $[0, t_1]$.

4) In the final main section of the paper, we derive sufficient conditions for the blow-up of the classical solution to the Cauchy problem (1), (2) in finite time.

For the reader's convenience, we also refer to a related article [7].

2. The Cauchy Problem for the Linear Homogeneous Equation

Let us consider the linear homogeneous equation corresponding to (1):

$$u_{tt} - a^{-1}u_{ttxx} + b(a^{-1}u_{xxxx} + 2u_{xx} + au) = 0. \quad (6)$$

We introduce a new unknown function:

$$w(t, x) = u(t, x) - a^{-1}u_{xx}(t, x), \quad (7)$$

assuming that the partial derivatives u_{xx} and u_{ttx} are continuous for $t \in \overline{\mathbb{R}}_+$. Under the assumption that the initial functions $\varphi(x)$, $\psi(x)$ belong to $C^{(2)}[\mathbb{R}]$, we can uniquely define the initial values of $w(t, x)$ as:

$$w|_{t=0} = w_0(x) = \varphi(x) - a^{-1}\varphi''(x), \quad w_t|_{t=0} = w_1(x) = \psi(x) - a^{-1}\psi''(x).$$

Using the fact that the positive real axis belongs to the resolvent set of the differential operator ∂_x^2 , we express the solution $u(t, x)$ of equation (6) via the new function $w(t, x)$ as:

$$u(t, x) = (I - a^{-1}\partial_x^2)^{-1} w(t, x) = \frac{\sqrt{a}}{2} \int_{-\infty}^{+\infty} e^{-|s|\sqrt{a}} w(t, x + s) ds. \quad (8)$$

Here, I denotes the identity operator.

As a result of substitution (8), we obtain the integro-differential equation equivalent to (6):

$$w_{tt} = (A_2 + A_1)w, \quad (9)$$

with operator coefficients defined by:

$$A_1 = -4a^2b(aI - \partial_x^2)^{-1}, \quad D(A_1) = C[\mathbb{R}]; \quad A_2 = b\partial_x^2 + 3ab, \quad D(A_2) = C^{(2)}[\mathbb{R}].$$

In the space $C[\mathbb{R}]$, the operator $b\partial_x^2$ generates a strongly continuous cosine operator function $C(\tau; b\partial_x^2)$, $\tau \in \mathbb{R}$, which is represented on elements $g(x) \in C[\mathbb{R}]$ as [8, § 1.5]:

$$C(\tau; b\partial_x^2)g(x) = \frac{1}{2} \left[g(x + \tau\sqrt{b}) + g(x - \tau\sqrt{b}) \right],$$

and satisfies the norm estimate:

$$\|C(t; b\partial_x^2)\| \leq 1, \quad t \in \overline{\mathbb{R}}_+.$$

The perturbed operator $A_2 = b\partial_x^2 + 3ab$ generates a strongly continuous cosine operator function $C(\tau; A_2)$, $\tau \in \mathbb{R}$, which for any $g(x) \in C[\mathbb{R}]$ has the representation [8, § 8.2]:

$$C(\tau; A_2)g(x) = C(\tau; b\partial_x^2)g(x) + \tau\sqrt{3ab} \int_0^\tau \frac{I_1\left(\sqrt{3ab(\tau^2 - s^2)}\right)}{\sqrt{\tau^2 - s^2}} C(s; b\partial_x^2)g(x) ds,$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind. The corresponding norm estimate is:

$$\|C(t; A_2)\| \leq \cosh(t\sqrt{3ab}), \quad t \in \overline{\mathbb{R}}_+.$$

The bounded operator $A_1 = -4a^2b(aI - \partial_x^2)^{-1}$, with $\|A_1\| \leq 4ab$, generates a cosine operator function $C(\tau; A_1)$ continuous in the uniform operator topology, which has the representation [8, § 1.4, § 4.2]:

$$C(\tau; A_1)g(x) = \sum_{n=0}^{\infty} \frac{\tau^{2n}}{(2n)!} A_1^n g(x) \quad (\forall g(x) \in C[\mathbb{R}]), \quad \tau \in \mathbb{R},$$

with the estimate:

$$\|C(t; A_1)\| \leq \cosh(2t\sqrt{ab}), \quad t \in \overline{\mathbb{R}}_+.$$

The operator $A = A_2 + A_1$ results from the bounded perturbation of the unbounded operator A_2 by A_1 . Since bounded perturbations preserve the cosine generator property [8, § 8.2], A generates a strongly continuous cosine operator function $C(\tau; A)$, $\tau \in \mathbb{R}$, which satisfies the representation for $g(x) \in C[\mathbb{R}]$:

$$C(\tau; A)g(x) = C(\tau; A_2)g(x) + \frac{\tau^2}{2} \int_0^1 j_1(\tau\sqrt{1-s^2}, A_1) A_1 C(\tau s; A_2)g(x) ds,$$

where

$$j_1(\tau, A_1)g(x) = \frac{4}{\pi} \int_0^1 \sqrt{1-r^2} C(\tau r; A_1)g(x) dr.$$

For $t \in \overline{\mathbb{R}}_+$, the following norm estimates hold:

$$\|j_1(t, A_1)\| \leq \frac{1}{t\sqrt{ab}} I_1(2t\sqrt{ab}),$$

$$\|C(t; A)\| \leq \cosh(t\sqrt{3ab}) + \frac{1}{2} \cosh(2t\sqrt{7ab}) =: h_1(t). \quad (10)$$

Associated with the cosine function $C(\tau; A)$ is the sine operator function [8, § 1.4]:

$$S(\tau; A)g(x) = \int_0^\tau C(s; A)g(x) ds, \quad g(x) \in C[\mathbb{R}],$$

and the linear manifold

$$C_1[\mathbb{R}] = \{g(x) \in C[\mathbb{R}] : C(\tau; A)g(x) \in C^1(\mathbb{R}, C[\mathbb{R}])\}.$$

That is, $C_1[\mathbb{R}] \subset C[\mathbb{R}]$ consists of those functions for which $C(\tau; A)g(x)$ is continuously differentiable in τ . Clearly, $\mathcal{D}(A) = C^{(2)}[\mathbb{R}] \subset C_1[\mathbb{R}]$.

Using estimate (10), we obtain

$$\|S(t; A)\| \leq \frac{\sinh(t\sqrt{3ab})}{\sqrt{3ab}} + \frac{\sinh(2t\sqrt{7ab})}{4\sqrt{7ab}} =: h_2(t), \quad t \in \overline{\mathbb{R}}_+. \quad (11)$$

Equation (9) can be rewritten in the space $C[\mathbb{R}]$ as an abstract second-order differential equation:

$$W_{tt} = AW, \quad t \in \mathbb{R}_+, \quad (12)$$

where $W = W(t) : t \rightarrow w(t, x)$ is the desired vector-valued function defined for $t \in \overline{\mathbb{R}}_+$ with values in $C[\mathbb{R}]$.

For equation (12), we consider the abstract Cauchy problem with initial conditions:

$$W|_{t=0} = W_0, \quad W_t|_{t=0} = W_1, \quad (13)$$

where $W_0 = w_0(x)$, $W_1 = w_1(x)$ are elements of the space $C[\mathbb{R}]$.

The Cauchy problem (12), (13) is uniformly well-posed [8, § 1.4] only when the operator A is the generator of a strongly continuous cosine operator function $C(t; A)$. In this case, the solution to the abstract Cauchy problem is given by the formula $W(t) = C(t; A)W_0 + S(t; A)W_1$ for any $W_0 \in \mathcal{D}(A)$ and $W_1 \in C_1[\mathbb{R}]$ [8, § 1.4].

Now, by applying the inverse transformation (8) and taking into account the commutativity of the resolvent $(I - a^{-1}\partial_x^2)^{-1}$ with the cosine operator function $C(t; A)$, we obtain the solution to the Cauchy problem (6), (2):

$$u(t, x) = C(t; A)\varphi(x) + S(t; A)\psi(x), \quad t \in \overline{\mathbb{R}}_+. \quad (14)$$

Thus, we have

Lemma 1. *Let the initial functions $\varphi(x)$, $\psi(x)$ belong to $C^{(4)}[\mathbb{R}]$. Then the Cauchy problem (6), (2) is uniformly well-posed, its classical solution is given by formula (14), and the following estimate holds:*

$$\sup_{x \in \mathbb{R}} |u(t, x)| \leq h_1(t) \sup_{x \in \mathbb{R}} |\varphi(x)| + h_2(t) \sup_{x \in \mathbb{R}} |\psi(x)|,$$

where the functions $h_1(t)$ and $h_2(t)$ are defined by inequalities (10) and (11).

REMARK 1. The classical solution $W(t)$ of the abstract Cauchy problem (12), (13) belongs to $C^{(2)}(\overline{\mathbb{R}}_+, C[\mathbb{R}])$ and satisfies $AW(t) \in C(\overline{\mathbb{R}}_+, C[\mathbb{R}])$; therefore, the solution $u(t, x)$ of equation (6) belongs to $C^{2,4}(\overline{\mathbb{R}}_+, \mathbb{R})$.

3. Local Solution of the Cauchy Problem for the Nonlinear Equation (4)

Equation (4) is obtained from equation (1) by differentiating both sides with respect to x and making the substitution $u_x = v$ (the left-hand sides of these equations coincide).

Applying the operator $(I - a^{-1}\partial_x^2)^{-1}$ to both sides of equation (4), we obtain an equivalent integro-differential equation, which can be written in the space $C[\mathbb{R}]$ as the following abstract semilinear equation:

$$V_{tt} = AV + F(V), \quad (15)$$

where $V = V(t) : t \mapsto v(t, x)$ is the sought vector-valued function, and the operator A is as in equation (12).

The nonlinear operator $F(\cdot)$ in equation (15) is defined by

$$F(g) = a [a(aI - \partial_x^2)^{-1} - I] f(g),$$

where $f(\cdot)$ is the Nemytskii (superposition) operator: $f(g) = f(g(x))$, $g(x) \in C[\mathbb{R}]$, and satisfies the norm estimate:

$$\|F(g)\|_C \leq 2a |f(\|g\|_C)|, \quad g \in C[\mathbb{R}]. \quad (16)$$

For equation (15), we consider the abstract Cauchy problem with initial conditions:

$$V|_{t=0} = V_0, \quad V_t|_{t=0} = V_1, \quad (17)$$

where $V_0 = \varphi'(x)$, $V_1 = \psi'(x)$ are elements of $C[\mathbb{R}]$.

The continuous differentiability of the superposition operator in the space of continuous functions, and the boundedness of the linear operator $(aI - \partial_x^2)^{-1}$, imply the Fréchet differentiability of $F(\cdot)$ in $C[\mathbb{R}]$. Therefore, there exists an interval $[0, t_0)$ on which the abstract Cauchy problem (15), (17) has [9, § 3] a unique classical solution $V = V(t)$ that satisfies the abstract integral equation

$$V(t) = C(t; A)V_0 + S(t; A)V_1 + \int_0^t S(t - \tau; A)F(V(\tau)) d\tau, \quad (18)$$

for any $V_0 \in \mathcal{D}(A)$ and $V_1 \in C_1[\mathbb{R}]$.

From the integral equation (18), using the estimates (10), (11), (16), and (3), we derive the integral inequality

$$\|V(t)\|_C \leq h_1(t)\|V_0\|_C + h_2(t)\|V_1\|_C + 2a \int_0^t h_2(t - \tau) |f(\|V(\tau)\|_C)| d\tau,$$

where $\|V_0\|_C = \sup_{x \in \mathbb{R}} |\varphi'(x)|$, $\|V_1\|_C = \sup_{x \in \mathbb{R}} |\psi'(x)|$.

Let

$$k_1 = \max_{t \in [0, t_0]} (h_1(t)\|V_0\|_C + h_2(t)\|V_1\|_C), \quad k_2 = 2a \max_{t \in [0, t_0]} h_2(t),$$

and increasing the right-hand side of the integral inequality, we have

$$\|V(t)\|_C \leq k_1 + k_2 \int_0^t |f(\|V(\tau)\|_C)| d\tau. \quad (19)$$

From inequality (19), we obtain [10, Chapter 1, § 1.4] an estimate for the norm of the classical solution of the abstract Cauchy problem (15), (17) in the space $C[\mathbb{R}]$ on the interval $[0, t_1]$:

$$\|V(t)\|_C \leq G^{-1}(G(k_1) + k_2 t) =: h_3(t),$$

where $G(t) = \int_\tau^t \frac{ds}{|f(s)|}$ for $\tau, t > 0$, $G^{-1}(\cdot)$ is the inverse function to $G(\cdot)$, and the interval $[0, t_1] \subset [0, t_0)$ is determined by those t for which $G(k_1) + k_2 t$ lies in the domain of definition of $G^{-1}(\cdot)$.

Thus, we have

Lemma 2. *Let the function $f(\cdot)$ satisfy condition (3), and let the initial functions $\varphi(x)$ and $\psi(x)$ belong to $C[\mathbb{R}]$ together with their derivatives up to order five inclusive. Then on the interval $[0, t_1]$, there exists a unique classical solution $v = v(t, x)$ to the Cauchy problem (4), (5), for which the following estimate holds:*

$$\sup_{x \in \mathbb{R}} |v(t, x)| = \sup_{x \in \mathbb{R}} |u_x(t, x)| \leq h_3(t), \quad t \in [0, t_1]. \quad (20)$$

4. Connection Between Solutions of Equations (1) and (4)

Assume that the classical solution $v = v(t, x) = u_x(t, x)$ of the auxiliary Cauchy problem (4), (5), along with its partial derivatives $v_{tt}(t, x)$, $v_{ttxx}(t, x)$, $v_{xxxx}(t, x)$, $v_{xx}(t, x)$, and the functions $v_x^2(t, x) \cdot f'(v(t, x))$ and $v_{xx}(t, x) \cdot f''(v(t, x))$, for all values of the time variable $t \in [0, t_1]$, belong (with respect to x) to the intersection of the space $C[\mathbb{R}]$ with the space $L_1(\mathbb{R})$ of absolutely integrable functions, i. e., $\int_{-\infty}^{+\infty} |g(x)| dx < \infty$. All these requirements are satisfied if on the time interval $t \in [0, t_1]$ the following conditions hold:

$$u_x, u_{ttx}, u_{ttxxx}, u_{xxxxx}, u_{xxx}, u_{xx}^2 \cdot f'(u_x), u_{xxx} \cdot f''(u_x) \in C[\mathbb{R}] \cap L_1(\mathbb{R}). \quad (21)$$

Note that from the fact that $g(x) \in C[\mathbb{R}] \cap L_1(\mathbb{R})$, i.e., the integral $\int_{-\infty}^{+\infty} |g(x)| dx < \infty$ converges and the limits $\lim_{x \rightarrow \pm\infty} g(x)$ exist, it follows [11, Ch. 2, §29.1] that these limits are equal to zero: $\lim_{x \rightarrow \pm\infty} g(x) = 0$.

Lemma 3. *The existence of a local classical solution $v = v(t, x)$, $t \in [0, t_1]$, to the Cauchy problem (4), (5) implies the existence of the corresponding solution*

$$u = u(t, x) = \lim_{x_0 \rightarrow -\infty} \int_{x_0}^x v(t, s) ds = \int_{-\infty}^x v(t, s) ds \quad (22)$$

to the Cauchy problem (1), (2) on the same time interval $[0, t_1]$, provided conditions (21) and the following limiting relations hold:

$$u_{tt}, u_{ttxx}, u_{xxxx}, u_{xx}, u \rightarrow 0 \quad \text{as } x \rightarrow -\infty, \quad t \in [0, t_1]. \quad (23)$$

◁ Let $v = v(t, x) = u_x(t, x)$, $t \in [0, t_1]$, be a classical solution of equation (4). Then using the limiting relations (23), we obtain:

$$\begin{aligned} \int_{-\infty}^x \partial_s^2 f(v(t, s)) ds &= \int_{-\infty}^x [u_{ss}^2(t, s) f'(u_s(t, s)) + u_{sss}(t, s) f''(u_s(t, s))] ds \\ &= u_{xx}(t, x) f'(u_x(t, x)) - \lim_{x_0 \rightarrow -\infty} u_{xx}(t, x_0) f'(u_x(t, x_0)) = u_{xx}(t, x) f'(u_x(t, x)). \end{aligned}$$

Now, substituting expression (22) into equation (1), we obtain an identity on the interval $[0, t_1]$, which implies that the function (22) is a solution of equation (1). ▷

5. Blow-Up of the Classical Solution to the Cauchy Problem (1), (2)

In the subsequent analysis, it is assumed that the classical solution $u = u(t, x)$, $(t, x) \in [0, t_1] \times \mathbb{R}$, of the Cauchy problem (1), (2), together with the partial derivatives appearing in equation (1), belongs to the intersection of the space $C[\mathbb{R}]$ and the space $L_2(\mathbb{R})$ of square-integrable functions. All these requirements are satisfied if, on the time interval $t \in [0, t_1]$, the following conditions hold:

$$\partial_t^n \partial_x^m u(t, x) \in C[\mathbb{R}] \cap L_2(\mathbb{R}), \quad n + m \leq 4, \quad n = 0, 1, 2, \quad m = 0, 1, 2, 3, 4. \quad (24)$$

Such a solution will be called a *classical solution of the Cauchy problem (1), (2) in the space $L_2(\mathbb{R})$* .

Note that from conditions (24) it follows that the following limiting relations hold:

$$\lim_{x \rightarrow \pm\infty} \partial_t^n \partial_x^m u(t, x) = 0, \quad n + m \leq 4, \quad n = 0, 1, 2, \quad m = 0, 1, 2, 3, 4, \quad t \in [0, t_1]. \quad (25)$$

Recall that the inner product and norm in $L_2(R)$ are defined by the formulas:

$$(\varphi, \psi) = \int_{-\infty}^{+\infty} \varphi(x) \psi(x) dx, \quad \|\varphi\|_2^2 = \int_{-\infty}^{+\infty} |\varphi(x)|^2 dx.$$

On the time interval $t \in [0, t_1]$ for which a classical solution $u = u(t, x)$ of the Cauchy problem (1), (2) exists, we introduce the functional:

$$y(t) = (u, u) + a^{-1}(u_x, u_x) = \int_{-\infty}^{+\infty} (u^2 + a^{-1}u_x^2) dx, \quad t \in [0, t_1].$$

Applying the derivative to the functional $y(t)$ gives:

$$y'(t) = 2((u, u_t) + a^{-1}(u_x, u_{tx})),$$

the Cauchy–Schwarz inequality: $|(\varphi, \psi)| \leq \|\varphi\|_2 \|\psi\|_2$, and, denoting

$$z(t) = (u_t, u_t) + a^{-1}(u_{tx}, u_{tx}) = \int_{-\infty}^{+\infty} (u_t^2 + a^{-1}u_{tx}^2) dx, \quad t \in [0, t_1],$$

which is another functional related to equation (1), we obtain the auxiliary estimate:

$$(y'(t))^2 \leq 4y(t)z(t). \quad (26)$$

We now seek sufficient conditions for the blow-up of the solution of the Cauchy problem (1), (2), meaning the appearance of a discontinuity of the second kind for the functional $y(t)$ on some interval $[0, t_2] \subseteq [0, t_1]$, which is chosen such that $y(t) > 0$ and $y'(t) \geq 0$ hold, resulting from the initial conditions $y(0) > 0$ and $y'(0) > 0$.

Theorem. Suppose the conditions of Lemmas 2 and 3 are satisfied and that the initial functions $\varphi(x)$ and $\psi(x)$ and the nonlinearity $f(\cdot)$ meet the following requirements:

$$\begin{aligned} & \|\varphi\|_2^2 + a^{-1}\|\varphi'\|_2^2 > 0, \quad (\varphi, \psi) + a^{-1}(\varphi', \psi') > 0, \\ & 2b\|\varphi'\|_2^2 + \|\psi\|_2^2 + a^{-1}\|\psi'\|_2^2 + (f'(\varphi')\varphi'', \varphi) > a^{-1}b\|\varphi''\|_2^2 + ab\|\varphi\|_2^2, \\ & F(x) = \int_0^x f(s) ds \geq 0 \quad (\forall x \in R); \quad F(u_x) \in L_1(R), \end{aligned}$$

$$\|\psi\|_2^2 + a^{-1}\|\psi'\|_2^2 + a^{-1}b\|\varphi''\|_2^2 + ab\|\varphi\|_2^2 + 2 \int_{-\infty}^{+\infty} F(\varphi'(x)) dx \geq 2b\|\varphi'\|_2^2,$$

$$\begin{aligned} & ((\varphi, \psi) + a^{-1}(\varphi', \psi'))^2 > (\|\varphi\|_2^2 + a^{-1}\|\varphi'\|_2^2) \\ & \quad \times \left(b \frac{2b(a + 2b) + 2a + k_3}{ak_3} \left(\|\varphi\|_2^2 + \frac{1}{a}\|\varphi'\|_2^2 \right) + E_0 \right), \end{aligned}$$

then the classical solution $u = u(t, x)$ of the Cauchy problem (1), (2) in the space $L_2(\mathbb{R})$ blows up in finite time

$$T_\infty \leq k_7^{-1} (\|\varphi\|_2^2 + a^{-1}\|\varphi'\|_2^2)^{-\frac{ak_3}{4b}},$$

and moreover, the following lower bound holds for the functional:

$$y(t) = \|u\|_2^2 + a^{-1}\|u_x\|_2^2 \geq \left((\|\varphi\|_2^2 + a^{-1}\|\varphi'\|_2^2)^{-\frac{ak_3}{4b}} - k_7 t \right)^{-\frac{4b}{ak_3}},$$

where the constants k_i and E_0 are determined by the parameters a, b , the initial functions $\varphi(x), \psi(x)$, and the nonlinearity $f(\cdot)$.

◁ Multiply both sides of equation (1) by $u = u(t, x)$ and integrate the resulting equality over x from $-\infty$ to $+\infty$. Then, integrating by parts and using the limiting relations (25), we obtain

$$(u_{tt}, u) + a^{-1}(u_{ttx}, u_x) + a^{-1}b(u_{xx}, u_{xx}) - 2b(u_x, u_x) + ab(u, u) - (f'(u_x)u_{xx}, u) = 0. \quad (27)$$

Using the representations

$$(u_{tt}, u) = \frac{d}{dt}(u_t, u) - (u_t, u_t) = \frac{1}{2} \frac{d^2}{dt^2} \|u\|_2^2 - \|u_t\|_2^2,$$

$$(u_{ttx}, u_x) = \frac{d}{dt}(u_{tx}, u_x) - (u_{tx}, u_{tx}) = \frac{1}{2} \frac{d^2}{dt^2} \|u_x\|_2^2 - \|u_{tx}\|_2^2,$$

and

$$(u_{tt}, u) + a^{-1}(u_{ttx}, u_x) = \frac{1}{2} y''(t) - z(t),$$

from (27) we derive the equality

$$y''(t) + 2a^{-1}b\|u_{xx}\|_2^2 + 2ab\|u\|_2^2 - 2(f'(u_x)u_{xx}, u) = 4b\|u_x\|_2^2 + 2z(t). \quad (28)$$

From equation (28) we find the value at zero of the second derivative of the functional $y(t)$, and require the following condition to hold:

$$2b\|\varphi'\|_2^2 + \|\psi\|_2^2 + a^{-1}\|\psi'\|_2^2 + (f'(\varphi')\varphi'', \varphi) - a^{-1}b\|\varphi''\|_2^2 - ab\|\varphi\|_2^2 > 0,$$

then on some interval $[0, t_3] \subseteq [0, t_2]$ the inequality $y''(t) \geq 0$ will be satisfied.

Applying inequalities (3) and (20), we estimate the fourth term on the left-hand side of equality (28):

$$\begin{aligned} -2(f'(u_x)u_{xx}, u) &\leq 2 \int_{-\infty}^{+\infty} |f'(u_x)| |u_{xx}| |u| dx \leq 2 \left| f' \left(\sup_{x \in \mathbb{R}} |u_x| \right) \right| (\|u_{xx}\|_2 \|u\|_2) \\ &\quad \text{(using the Cauchy-Schwarz inequality)} \end{aligned} \quad (29)$$

$$\leq 2|f'(h_3(t))| \|u_{xx}\|_2 \|u\|_2 \leq k_3 (\|u_{xx}\|_2^2 + \|u\|_2^2),$$

where $k_3 = \max_{t \in [0, t_1]} |f'(h_3(t))|$.

Taking into account the sequence of inequalities (29), we increase the left-hand side of equality (28) while simultaneously decreasing the right-hand side:

$$y''(t) + (2a^{-1}b + k_3)\|u_{xx}\|_2^2 + (2ab + k_3)\|u\|_2^2 \geq 2z(t), \quad t \in [0, t_3]. \quad (30)$$

Multiply both sides of equation (1) by $u_t = u_t(t, x)$ and integrate the resulting equality over x from $-\infty$ to $+\infty$. Then, integrating by parts, we have

$$(u_{tt}, u_t) + a^{-1}(u_{ttx}, u_{tx}) + a^{-1}b(u_{xx}, u_{txx}) - 2b(u_x, u_{tx}) + ab(u, u_t) + (f(u_x), u_{tx}) = 0. \quad (31)$$

Denote by $F(\cdot)$ the function determined by the nonlinearity of equation (1):

$$F(x) = \int_0^x f(s) ds,$$

and assuming $F(u_x) \in L_1(R)$, $t \in [0, t_3]$, from equality (31) we obtain

$$\frac{1}{2} \frac{d}{dt} E(t) = 0, \quad (32)$$

where $E(t)$ is the functional associated with equation (1):

$$E(t) = \|u_t\|_2^2 + a^{-1}\|u_{tx}\|_2^2 + a^{-1}b\|u_{xx}\|_2^2 - 2b\|u_x\|_2^2 + ab\|u\|_2^2 + 2 \int_{-\infty}^{+\infty} F(u_x) dx.$$

From equality (32) it follows that $E(t)$ does not depend on time, and therefore the equality holds

$$E(t) = E(0) = E_0, \quad (33)$$

where the constant E_0 is defined by the formula

$$E_0 = \|\psi\|_2^2 + a^{-1}\|\psi'\|_2^2 + a^{-1}b\|\varphi''\|_2^2 - 2b\|\varphi'\|_2^2 + ab\|\varphi\|_2^2 + 2 \int_{-\infty}^{+\infty} F(\varphi'(x)) dx.$$

Let $E_0 \geq 0$, and let the nonlinearity of equation (1) satisfy the condition

$$F(x) = \int_0^x f(s) ds \geq 0 \quad (\forall x \in R). \quad (34)$$

Note that the set of functions $f(\cdot)$ satisfying condition (34) is not empty. Indeed, for example, the functions $f(x) = x^{2n+1}$, where $n > 0$, satisfy condition (34).

Rewrite equality (33) as

$$z(t) + a^{-1}b\|u_{xx}\|_2^2 + (a+2)b\|u\|_2^2 + 2 \int_{-\infty}^{+\infty} F(u_x) dx = E_0 + 2by(t). \quad (35)$$

Decreasing the left-hand side of equality (35), we derive the estimate

$$\|u_{xx}\|_2^2 \leq ab^{-1}(E_0 + 2by(t) - z(t)). \quad (36)$$

Using inequality (26), we decrease the right-hand side, and applying inequalities (36) and $\|u\|_2^2 \leq y(t)$, we increase the left-hand side of inequality (30). Then we obtain the differential inequality

$$y(t)y''(t) - k_4(y'(t))^2 + k_5y^2(t) + k_6y(t) \geq 0, \quad t \in [0, t_3], \quad (37)$$

where the coefficients are computed by the formulas

$$k_4 = 1 + \frac{ak_3}{4b}, \quad k_5 = 2b(a+2) + (2a+1)k_3, \quad k_6 = \left(2 + \frac{ak_3}{b}\right) E_0,$$

and we note that $k_4 > 1$ and $\delta = 1 - \frac{k_4}{1-k_4} = 2\left(1 + \frac{2b}{ak_3}\right) > 0$.

Let the initial conditions hold

$$y'(0) > \left(\frac{k_5}{k_4 - 1} (y(0))^2 + \frac{2k_6}{(k_4 - 1)\delta} y(0) \right)^{\frac{1}{2}} = 2 \left(b \frac{2b(a+2) + (2a+1)k_3}{ak_3} (y(0))^2 + E_0 y(0) \right)^{\frac{1}{2}}$$

or, substituting the values of $y(0)$ and $y'(0)$,

$$\left((\varphi, \psi) + \frac{1}{a} (\varphi', \psi') \right)^2 > b \frac{2b(a+2) + (2a+1)k_3}{ak_3} \left(\|\varphi\|_2^2 + \frac{1}{a} \|\varphi'\|_2^2 \right)^2 + E_0 \left(\|\varphi\|_2^2 + \frac{1}{a} \|\varphi'\|_2^2 \right).$$

Then, comparing inequality (37) with an ordinary differential inequality [12, Appendix, § 2, (2.38)], we conclude that for the time t_3 of existence of the solution to the Cauchy problem (1), (2), the following upper estimate holds

$$t_3 < T_\infty \leq \frac{1}{k_7 (y(0))^{k_4-1}} = \frac{1}{k_7 \left(\|\varphi\|_2^2 + a^{-1} \|\varphi'\|_2^2 \right)^{\frac{ak_3}{4b}}},$$

where

$$\begin{aligned} k_7^2 &= \frac{(k_4 - 1)^2}{(y(0))^{2k_4}} \left[(y'(0))^2 - \frac{k_5}{k_4 - 1} (y(0))^2 - \frac{2k_6}{(k_4 - 1)\delta} (y(0))^{\delta(1-k_4)+2k_4} \right] \\ &= \frac{a^2 k_3^2}{4b^2} \left(\|\varphi\|_2^2 + a^{-1} \|\varphi'\|_2^2 \right)^{-2 - \frac{ak_3}{2b}} \left[\left((\varphi, \psi) + a^{-1} (\varphi', \psi') \right)^2 \right. \\ &\quad \left. - b \frac{2b(a+2b) + 2a + k_3}{ak_3} \left(\|\varphi\|_2^2 + a^{-1} \|\varphi'\|_2^2 \right)^2 - E_0 \left(\|\varphi\|_2^2 + a^{-1} \|\varphi'\|_2^2 \right) \right], \end{aligned}$$

and for the functional $y(t) = \|u\|_2^2 + a^{-1} \|u_x\|_2^2$ the following lower bound holds

$$y(t) \geq \frac{1}{((y(0))^{1-k_4} - k_7 t)^{\frac{1}{k_4-1}}} = \frac{1}{\left(\left(\|\varphi\|_2^2 + a^{-1} \|\varphi'\|_2^2 \right)^{-\frac{ak_3}{4b}} - k_7 t \right)^{\frac{4b}{ak_3}}}$$

and, therefore,

$$\lim_{t \uparrow T_\infty} y(t) = +\infty.$$

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РАЗРУШЕНИЕ РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ УРАВНЕНИЯ МАЛЫХ КОЛЕБАНИЙ АБСОЛЮТНО ГИБКОГО ОДНОРОДНОГО СТЕРЖНЯ

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Аннотация. В данной работе исследуется задача Коши для нелинейного дифференциального уравнения в частных производных типа Соболева (т. е. уравнения, не разрешенного относительно старшей производной по времени), обобщающего уравнение малых колебаний абсолютно гибкого однородного стержня. Задача рассматривается в пространстве непрерывных функций, определенных на всей вещественной оси, для которых существуют пределы на бесконечности. Получена явная форма классического решения соответствующего линейного однородного уравнения, а также выведены оценки нормы для операторнозначных функций, представляющих это решение. Для задачи Коши линейного однородного уравнения также получена оценка нормы решения. Установлен временной интервал существования и

единственности классического решения вспомогательной задачи Коши, связанной с исходной, а также оценка нормы этого локального решения. Найдены условия, обеспечивающие соответствие между классическими решениями исходной и вспомогательной задач Коши на определенном временном интервале. Кроме того, установлены условия, при которых классическое решение задачи Коши разрушается за конечное время.

Ключевые слова: малые колебания абсолютно гибкого однородного стержня, нелинейное уравнение типа Соболева, взрыв решения.

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