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CHARACTERISTIC CAUCHY PROBLEM FOR A LOADED WAVE EQUATION

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Abstract. The paper considers a loaded wave equation. Its peculiarity is that in characteristic variables $\xi = x - y$, $\eta = x + y$, the loaded components are a linear combination of unknown functions, which depend only on ξ or η . Moreover, the loaded part of this equation is a solution to the wave equation. For such an equation a representation of all its solutions is obtained. The problem is investigated in the case where the carrier is one of the characteristics of the wave equation, a setting known to be ill-posed for the usual string oscillation equation. For the Cauchy problem, the same characteristic carrier is considered. For a normal wave equation, such a formulation is known to be ill-posed. When some nonlocal conditions of a point nature are satisfied, the existence and uniqueness theorem for the given problem is proved, and the solution is written out explicitly. This shows the effect of the load as a regulator of the ill-posed problem.

Keywords: loaded equation, wave equation, wave operator, d'Alembert formula, characteristic Cauchy problem.

AMS Subject Classification: 35L05, 35L10, 35L20.

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1. Introduction

Consider the equation

$$\square u(x, y) = 4[\lambda u(x - y, 0) + \mu u(x + y, 0)], \quad (1)$$

where

$$\square = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \quad (2)$$

is the wave operator, and λ and μ are real constants.

In less than fifty years since the publication of the work [1], in which the definition of loaded equations was first introduced, a significant number of publications focusing on loaded differential equations have emerged. Note only some of them [2–5].

It is important to know that the loaded differential equation in the domain $D \in \mathbb{R}^n$ has the form

$$Lu(z) + Mu(z) = f(z), \quad (3)$$

where L is a differential operator with respect to $z \in D$, and M is an operator containing traces of operations from the function $u(z)$ on a manifold D_M of dimension strictly less than n ,

$D_M \subset \overline{D}$ (as usual, $\overline{D}$ denotes the closure of D). The term $Mu(z)$ is called the *load* in (3), and the manifold D_M is usually called the *trace* of the load [1–3].

The main feature of (1) is that its load $\lambda u(x - y, 0) + \mu u(x + y, 0)$ is a linear combination of the values of the sought solution at the intersection points of the line $y = 0$ with the characteristics of the wave equation emitted from the point (x, y) . Such a load is usually called *characteristic load*. The works [1, 6, 7] are devoted to the study of the Darboux and Goursat problem for strictly and weakly hyperbolic equations with characteristic loads.

The presence of a load in a differential equation can significantly affect the formulation and correctness of problems. This is especially evident in the case of hyperbolic equations. For example, the load traces of two model equations

$$u_x - u_y = \lambda u(x, 0), \quad u_x - u_y = \lambda u(x - y, 0), \quad \text{and} \quad u_x - u_y = \lambda u(x + y, 0),$$

are the same: they lie on the x -axis. However, their general solutions

$$u(x, y) = f(x + y) + \lambda \int_0^x e^{\lambda(x-t)} f(t) dt,$$

$$u(x, y) = f(x + y) - \frac{\lambda}{2} \int_{x-y}^{x+y} f(t) dt$$

and

$$u(x, y) = (1 - \lambda y) f(x + y)$$

are fundamentally different.

As is known, the Cauchy problem plays an important role in describing all solutions of partial differential equations [8, p. 73]. The Cauchy problem and control problems for loaded hyperbolic equations of the first and second orders with load traces located on a line parallel to the time axis (i. e., with spatially nonlocal loads) were solved in [9–11]. The Cauchy problem for a class of nonlinear first-order partial differential equations with load traces located on a finite union of straight lines parallel to the spatial axis (i. e., with loads nonlocal in time) was considered in [12–14].

For the classical one-dimensional wave equation

$$\square u(x, y) = 0, \tag{4}$$

the fundamental role in the study of initial-boundary value problems and their physical interpretation, is played by the d'Alembert formula,

$$u(x, y) = f(x - y) + g(x + y), \tag{5}$$

which gives the general representation for solutions of the equation (4).

It is known that for (4), the initial data given on the characteristic line cannot be chosen arbitrarily. They must satisfy some compatibility condition. Moreover, the corresponding homogeneous Cauchy problem with data given on one of the characteristics, say, on the line $x + y = 0$, has an infinite number of non-trivial solutions of the form $u(x, y) = F(x + y)$, where $F(t)$ is an arbitrary sufficiently smooth function satisfying $F(0) = F'(0) = 0$.

In this work, we study the Cauchy problem for the equation (1), namely, in the case when the Cauchy data are specified on the characteristic line, i. e., we study the characteristic

Cauchy problem. We show that the loaded term in the equation (1) plays a regularizing role: the characteristic Cauchy problem turns out to be correct, in contrast to the case of the equation (4). We find an analogue of the formula (5) for the equation (1), prove the existence and uniqueness theorem of solutions for the problem under consideration, and construct an explicit representation for the solutions.

2. Statement of the Problem

Let Ω be a domain in the plane of x and y , bounded by the segments of characteristics $AC : x - y = 0$, $AD : x + y = 0$, $BC : x + y = 1$, $BD : x - y = 1$ of the operator (2), i. e.,

$$\Omega = \{(x, y) : 0 < x - y < 1, 0 < x + y < 1\}.$$

DEFINITION 1. A function $u(x, y)$ is called a regular solution of the equation (1) in the domain Ω if $u(x, y) \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $u(x, y)$ satisfies (1) at any point $(x, y) \in \Omega$.

Let p be a given real number such that $0 \leq p \leq 1$. Let us denote the characteristic of the operator (2) passing through the points with coordinates $(p/2, p/2)$ and $((1+p)/2, (p-1)/2)$ by γ and the point with coordinates $(p/2, -p/2)$, $((1+p)/2, (1-p)/2)$ by γ_1 . Obviously, the point with coordinates $(p, 0)$ belongs to γ and to γ_1 .

PROBLEM 1: Find a regular solution of the equation (1) satisfying the conditions

$$u|_{\gamma} = \tau(x), \quad 0 \leq x \leq 1, \quad (6)$$

$$\frac{1}{\sqrt{2}} \frac{\partial u}{\partial \vec{n}} \Big|_{\gamma} = \nu(x), \quad 0 < x < 1, \quad (7)$$

where $\vec{n}$ is the normal to γ .

PROBLEM 2: Find a regular solution of the equation (1) satisfying the conditions

$$u|_{\gamma_1} = \tau(x), \quad 0 \leq x \leq 1, \quad (8)$$

$$\frac{1}{\sqrt{2}} \frac{\partial u}{\partial \vec{n}} \Big|_{\gamma_1} = \nu(x), \quad 0 < x < 1, \quad (9)$$

where $\vec{n}$ is the normal to γ_1 .

3. D'Alembert Formula

Proposition 1. Let $u(x, y)$ be a regular solution of the equation (1) in Ω . Then there exist functions $f(t)$ and $g(t)$ such that

$$\begin{aligned} u(x, y) = & f(x - y) + g(x + y) - \lambda(x + y) \int_{x-y}^{x+y} [f(t) + g(t)] dt \\ & + \mu(x - y) \int_{x-y}^{x+y} [f(t) + g(t)] dt. \end{aligned} \quad (10)$$

Conversely, a function of the form (10) is a regular solution of (1) if

$$f(t), g(t) \in C[0, 1] \cap C^2(0, 1). \quad (11)$$

◁ Let $u(x, y)$ be a regular solution of the equation (1), and let

$$\xi = x - y \quad \text{and} \quad \eta = x + y, \quad (12)$$

and

$$v(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\eta - \xi}{2}\right).$$

In terms of independent variables ξ and η , and the dependent variable $v(\xi, \eta)$, the equation (1) takes the form

$$v_{\xi\eta} = \lambda v(\xi, \xi) + \mu v(\eta, \eta). \quad (13)$$

Obviously, the transformation

$$V(\xi, \eta) = v(\xi, \eta) + \lambda \eta \int_{\xi}^{\eta} v(t, t) dt - \mu \xi \int_{\xi}^{\eta} v(t, t) dt \quad (14)$$

equivalently reduces (13) to the equation

$$V_{\xi\eta} = 0. \quad (15)$$

If we take $\xi = \eta$, then from (14), we obtain

$$V(\xi, \xi) = v(\xi, \xi) \quad (16)$$

Considering that the general solution (15)

$$V(\xi, \eta) = f(\xi) + g(\eta)$$

and taking into account (16), inverting (14), we get

$$v(\xi, \eta) = f(\xi) + g(\eta) - \lambda \eta \int_{\xi}^{\eta} [f(t) + g(t)] dt + \mu \xi \int_{\xi}^{\eta} [f(t) + g(t)] dt. \quad (17)$$

Returning to the variables x and y , we obtain the formula (10). This proves that a regular solution of the equation (1) can be represented in the form (10).

A simple direct calculation shows that for any functions $f(t)$ and $g(t)$ satisfying (11), the function (10) is a regular solution of the equation (1). ▷

4. Main Results

Theorem 1. Let $\tau(x) \in C^1[0, 1] \cap C^2(0, 1)$, $\nu(x) \in C^2[0, 1] \cap C^3(0, 1)$, and let $\lambda \neq 0$,

$$\nu'(p) = (\lambda + \mu)\tau(p), \quad \nu''(p) = \lambda[\tau'(p) + \nu(p)].$$

Then the solution to the Cauchy problem (6), (7) for equation (1) exists, is unique and has the form

$$\begin{aligned} u(x, y) = & \frac{1}{\lambda} \nu'(x + y) + \tau(x - y) - (x + y - p)[\nu(x + y) - \nu(x - y)] \\ & - \frac{2\mu}{\lambda} y[\nu(x + y) - \nu(p)] + \frac{\mu}{\lambda} [2(\lambda + \mu)(y^2 + xy - yp) - 1]\tau(p). \end{aligned} \quad (18)$$

◁ In terms of the variables (12), the conditions (6) and (7) take the form

$$v(\xi, p) = \tau(\xi) \quad \text{and} \quad v_\eta(\xi, p) = \nu(\xi), \quad (19)$$

and from (17) we obtain

$$f(\xi) + g(p) + (\mu\xi - \lambda p) \int_{\xi}^p [f(t) + g(t)] dt = \tau(\xi), \quad (20)$$

$$g'(p) - \lambda \int_{\xi}^p [f(t) + g(t)] dt + (\mu\xi - \lambda p)[f(p) + g(p)] = \nu(\xi). \quad (21)$$

Without loss of generality, we may assume that the normal $\vec{n}$ forms an acute angle with the Ox -axis.

Replacing ξ everywhere in (20) and (21) with p , we get

$$f(p) + g(p) = \tau(p), \quad g'(p) = \nu(p) + (\lambda - \mu)p\tau(p). \quad (22)$$

From (21), taking into account (22), we have

$$\int_{\xi}^p [f(t) + g(t)] dt = \frac{1}{\lambda} [\nu(p) + \mu(\xi - p)\tau(p) - \nu(\xi)] \quad (23)$$

From (20), taking into account (23) for $f(\xi)$, we get

$$f(\xi) = \tau(\xi) + \frac{1}{\lambda} (\mu\xi - \lambda p) [\nu(\xi) - \lambda\nu(p) - \mu(\xi - p)\tau(p)] - g(p). \quad (24)$$

From (23) after differentiation with respect to ξ , taking into account (24), for $g(\eta)$ we obtain

$$g(\eta) = \frac{1}{\lambda} [\nu'(\eta) - \mu\tau(p)] - \tau(\eta) - \frac{1}{\lambda} (\mu\eta - \lambda p) [\nu(\eta) - \nu(p) - \mu(\eta - p)\tau(p)] + g(p) \quad (25)$$

It also follows from (23) that

$$\int_{\xi}^{\eta} [f(t) + g(t)] dt = \frac{1}{\lambda} [\nu(\eta) - \nu(\xi) - \mu(\eta - \xi)\tau(p)]. \quad (26)$$

Substituting into (17) the expressions for $f(\xi)$, $g(\eta)$ and $\int_{\xi}^{\eta} [f(t) + g(t)] dt$, defined by formulas (24), (25) and (26), respectively, after some simple transformations, we obtain

$$\begin{aligned} v(\xi, \eta) = & \frac{1}{\lambda} \nu'(\eta) + \tau(\xi) - \tau(\eta) - (\eta - p)[\nu(\eta) - \nu(\xi)] \\ & - \frac{\mu}{\lambda} (\eta - \xi)[\nu(\eta) - \nu(p)] + \frac{\mu}{\lambda} [(\lambda + \mu)(\eta - \xi)(\eta - p) - 1]\tau(p). \end{aligned}$$

Returning to the variables x and y leads to (27). ▷

Theorem 2. Let $\tau(x) \in C^1[0, 1] \cap C^2(0, 1)$, $\nu(x) \in C^2[0, 1] \cap C^3(0, 1)$, and let $\mu \neq 0$,

$$\nu'(p) = (\lambda + \mu)\tau(p), \quad \nu''(p) = \mu[\tau'(p) + \nu(p)].$$

Then the solution to the Cauchy problem (8), (9) for equation (1) exists, is unique and has the form

$$u(x, y) = \frac{1}{\mu} \nu'(x - y) + \tau(x + y) - (x - y - p) [\nu(x - y) - \nu(x + y)] - \frac{2\lambda}{\mu} y [\nu(x - y) - \nu(p)] + \frac{\lambda}{\mu} [2(\lambda + \mu)(y^2 - xy + yp) - 1] \tau(p). \quad (27)$$

The proof of Theorem 2 is carried out in a completely similar way to the proof of Theorem 1.

REMARK 1. The condition $\lambda \neq 0$ is essential. Indeed, it is easy to see that the homogeneous Cauchy problem (6), (7) for (1) has an infinite number of solutions of the form

$$u(x, y) = f(x + y) + \mu(x - y) \int_p^{x+y} e^{\frac{\mu}{2}[(x+y)^2 - t^2]} f(t) dt,$$

where $f(\xi)$ is an arbitrary twice continuously differentiable function satisfying the condition $f(p) = f'(p) = 0$.

REMARK 2. Formula (27) implies that the domains of influence and definition, when the Cauchy data are specified on γ for the equation (1), coincide with the domains of influence and definition for the Cauchy problem for the wave equation (4) when the initial data are given on the segment $\{(x, y) : 0 \leq x \leq 1, y = 0\}$.

5. Conclusion

The solvability of Problem 1 in the case where $\square u$ is replaced by a strictly hyperbolic operator of general form with variable coefficients is equivalently reduced, using the Riemann method, to the solvability of a Volterra integral equation of the second kind, from which existence and uniqueness follow.

In this paper we show that loaded terms in hyperbolic equations can play the role of a regularizer. We present such an equation for which the characteristic Cauchy problem, i. e., the problem with data given on the characteristic lines, is well-posed, unlike the classical wave equation. In addition, this effect allows us to develop a theory of basic boundary value problems for equation (1), which significantly complements and expands the theory of boundary value problems for the one-dimensional wave equation.

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ХАРАКТЕРИСТИЧЕСКАЯ ЗАДАЧА КОШИ ДЛЯ НАГРУЖЕННОГО ВОЛНОВОГО УРАВНЕНИЯ

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Аннотация. В работе рассматривается одномерное нагруженное волновое уравнение. Особенностью такой нагрузки является то, что в характеристических переменных $\xi = x - y$, $\eta = x + y$ нагруженные компоненты представляют собой линейную комбинацию неизвестных функций, зависящих только от ξ или η . Для исходного волнового уравнения получено представление всех его решений, являющееся аналогом представления общего решения обычного уравнения колебания струны. Для него исследована задача, в которой носителем данных Коши является одна из характеристик рассматриваемого нагруженного волнового уравнения. Для обычного уравнения колебания струны такая задача является некорректной. При выполнении некоторых нелокальных условий точечной природы доказана теорема существования и единственности решения, а само оно выписано в явном виде. Это показывает влияние нагрузки как

регулятора для некорректной задачи. Регуляризация задачи Коши с данными на характеристиках происходит за счет наличия нагруженных слагаемых специального вида, каждое из которых отвечает за корректность на соответствующей характеристике.

Ключевые слова: нагруженное уравнение, волновое уравнение, волновой оператор, формула Даламбера, характеристическая задача Коши.

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