

УДК 517.957

DOI 10.46698/f6031-1077-3134-o

THE TURING–HOPF BIFURCATION PROBLEM
IN THREE-DIMENSIONAL REACTION-DIFFUSION SYSTEMS

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Abstract. The article discusses the issues of studying local bifurcations in the vicinity of spatially homogeneous equilibrium positions of the reaction-diffusion system in a limited area with homogeneous Neumann boundary conditions. The main results concern the interaction between the Turing bifurcation and the Andronov–Hopf bifurcation, which leads to the Turing–Hopf bifurcation. In such a situation, there are two types of diffusive instabilities, which can lead to a variety of effects, accompanied by the emergence of complex space-time regimes. The article discusses in detail the conditions for Turing–Hopf bifurcations and offers new necessary and sufficient criteria bifurcations, leading to effective formulas for studying them. When solving the main problem of the signs of the Turing–Hopf bifurcation, we obtain calculation formulas for studying Turing and Andronov–Hopf bifurcations in two-dimensional and three-dimensional systems; these formulas are of independent theoretical and practical interest.

Keywords: reaction-diffusion, equilibrium position, stability, bifurcation, Turing, Andronov–Hopf, Turing–Hopf.

AMS Subject Classification: 37G10, 35K57.

For citation: Vasenina, N. A. and Yumagulov, M. G. The Turing–Hopf Bifurcation Problem in Three-Dimensional Reaction-Diffusion Systems, *Vladikavkaz Math. J.*, 2026, vol. 28, no. 3, pp. 60–73. DOI: 10.46698/f6031-1077-3134-o.

1. Introduction

The study of reaction-diffusion systems has traditionally been the subject of a large number of works due to the wide range of its applications (see, for example, [1–4] and the existing bibliography). One of the most important areas of research in such systems is the study of critical phenomena and related bifurcations that lead to the appearance of equilibrium positions in the vicinity dissipative structures (Turing bifurcation) and autowave processes (Andronov–Hopf bifurcation). The emergence of such solutions is due to the fact that the eigenvalues of the corresponding linearized system cross the imaginary axis: through the value $\lambda = 0$ for dissipative structures and through the value $\lambda = i\omega$ (where $\omega \neq 0$) for self-waves.

There is extensive literature on the study of bifurcations in reaction-diffusion systems (see, for example, [1, 3–6]).

In recent years, there has been a significant increase in interest in the study of bifurcations with high codimension: Turing–Hopf, Turing–Turing, Hopf–Hopf, and other bifurcations. Such bifurcations are characterized by rich non-trivial dynamics, including stable and unstable

equilibrium states, spatially homogeneous and heterogeneous periodic solutions, and other complex spatio-temporal regimes.

Extensive literature of a theoretical and experimental nature is devoted to the study of these complex bifurcations (see, for example, [7–14]). At the same time, in different sources, bifurcations can be understood and studied in different settings. For example, the Turing–Hopf bifurcation can be considered as a situation where in a point system there is an Andronov–Hopf bifurcation, and in a distributed system there is a Turing bifurcation (see, for example, [1, 10, 14]). In other works (see, for example, [7, 8, 11, 12, 15]) it is assumed that both bifurcations (both Andronov–Hopf and Turing) are present only in the distributed system (at different wave numbers). The requirements may be different for the stability of the linearized point and/or distributed system at critical values of the parameter: in some works stability is assumed, in others there is no such requirement. Each of the considered formulations and the study of the corresponding problems are of both theoretical and practical interest.

It should also be noted that the term “Turing–Hopf bifurcation” itself is not generally accepted. Other terms are also used, such as “oscillatory Turing”, “pseudo-Turing” and others (see [1]). It seems to us more correct to speak of a “Turing–Andronov–Hopf bifurcation”, since this bifurcation is realized in a situation where, for a certain value of the parameters, the conditions of both the Turing bifurcation and the Andronov–Hopf bifurcation are satisfied in the system.

One of the important problems in the study of complex bifurcations is the problem of the signs of the corresponding bifurcation. The codimension of such bifurcations is usually equal to two or more, which requires the presence of an appropriate number of independent parameters for the effective study of the bifurcation. To solve this problem, an approach based on the construction and further investigation (theoretical and instrumental) of variance functions for the corresponding bifurcations is common (see, for example, [7, 8, 14]). As a rule, dispersion functions contain, along with the selected bifurcation parameters, a number of other parameters, which significantly complicates their study. It should also be noted that obtaining bifurcation signs is often carried out without a detailed study of the issues concerning the necessary conditions for the stability of the solutions arising at the bifurcation, namely, the issues on the stability of the linearized point and/or distributed system at critical values of the parameter.

2. Setting the Problem

In this paper, we study the problems of necessary and sufficient features of bifurcations of high codimension in dynamical systems of the type “reaction-diffusion”. For the sake of certainty, the Turing–Hopf bifurcation problem is discussed. The proposed research schemes can be modified for problems involving other types of high-codimension bifurcations (Turing–Turing, Hopf–Hopf, etc.). The paper provides a detailed discussion of the conditions for Turing–Hopf bifurcations and proposes new necessary and sufficient conditions for Turing–Hopf bifurcations, which lead to effective formulas for studying such bifurcations. When solving the main problem of the signs of the Turing–Hopf bifurcation, auxiliary questions are discussed about the signs of the Turing and Andronov–Hopf bifurcations; these formulas are of independent theoretical and practical interest. The obtained necessary and sufficient signs of the Turing–Hopf bifurcation include the necessary conditions for the stability of the linearized point and distributed systems.

The main object of research is the reaction-diffusion system, which is described by a

differential equation with Neumann boundary conditions (see, for example, [1–5])

$$\frac{dw}{dt} = \left(A(\mu) + D(\mu) \frac{\partial^2}{\partial x^2} \right) w + h(w, \mu), \quad w \in R^N, \quad (1)$$

$$\frac{\partial w}{\partial x}(0, t) = \frac{\partial w}{\partial x}(\pi, t) = 0, \quad (2)$$

where $A(\mu)$ and $D(\mu)$ are square (of order N) matrices whose elements depend smoothly (continuously differentiable) on the parameter μ , and the elements of the matrix $D(\mu)$ are non-negative. The smooth nonlinearity $h(w, \mu)$ satisfies the following condition uniformly in μ : $\|h(w, \mu)\| = O(\|w\|^2)$, $\|w\| \rightarrow 0$; here and below, $\|\cdot\|$ is the Euclidean norm in the space R^N .

The main purpose of this work is to study three-dimensional (three-component) systems (1)–(2), i.e., the case where $N = 3$. We note in this regard that three-component reaction-diffusion models and, accordingly, bifurcation problems in such models are a popular object of study due to their wide range of applications (see, for example, [7, 8, 20]). The work continues the research started in [16–19].

3. Basic Concepts and Designations

3.1. Solutions of the reaction-diffusion system. Let $\Omega = \{x : 0 \leq x \leq \pi\}$. We denote by $L_2(\Omega)$ the Hilbert space of functions $v(x)$ defined on the segment Ω , and by $W_2^2(\Omega)$ — Sobolev space with a norm

$$\|v(x)\|_{W_2^2} = \left(\int_{\Omega} \sum_{j=0}^2 \|v^{(j)}(x)\|^2 dx \right)^{\frac{1}{2}}.$$

Let $C(\Omega)$ and $C^2(\Omega)$ denote the spaces of continuous and twice continuously differentiable functions. We also define the set

$$C_0^2(\Omega) = \left\{ v \in C^2 : \frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial x}(\pi) = 0 \right\}. \quad (3)$$

The one-dimensional Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} : C_0^2 \rightarrow C$ can be (see, for example, [21]) extended to a closed self-adjoint operator $\Delta : L_2 \rightarrow L_2$ with domain G_0 formed by the closure in W_2^2 of the set $C_0^2(\Omega)$. The spectrum of the operator Δ consists of isolated eigenvalues $\lambda = -k^2$ of finite multiplicity (k are non-negative integers). *Solutions* of the system (1)–(2) will be called functions $w(x, t)$ that:

- for each fixed value of t , are elements of the space $W_2^2(\Omega)$;
- for each fixed value of $x \in \Omega$, are continuously differentiable functions with respect to t ;
- satisfy the equation (1) and the boundary conditions (2) for all $t \geq 0$ and $x \in \Omega$.

The system (1)–(2) has a spatially uniform equilibrium point $w = 0$.

3.2. Stability of equilibrium points. The equilibrium point $w = 0$ of the system (1)–(2) is called Lyapunov stable (see, for example, [4]), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0(x)\|_{W_2^2} < \delta$, then the solution $w(x, t)$ of the system (1)–(2) satisfies the inequality $\|w(x, t)\|_{W_2^2} < \varepsilon$ for all $t > 0$; here $w(x, t)$ is the solution of the Cauchy problem for the system (1)–(2) with the initial condition $w(x, 0) = u_0(x)$. If, in addition, $\|w(x, t)\|_{W_2^2} \rightarrow 0$ as $t \rightarrow \infty$, then the equilibrium point $w = 0$ is called *asymptotically stable*. The concept of an *unstable* equilibrium point $w = 0$ of the system (1)–(2) is naturally defined.

The stability of the solution $w = 0$ of the system (1)–(2) is determined by the properties of the linearized equation

$$\frac{dw}{dt} = (A(\mu) + D(\mu)\Delta)w, \quad w \in R^N,$$

namely, by the properties of the spectrum of the linear operator

$$S(\mu) = A(\mu) + D(\mu)\Delta : L_2(\Omega) \rightarrow L_2(\Omega), \quad (4)$$

with a dense in $L_2(\Omega)$ domain of definition G_0 , formed by the closure in W_2^2 of the set (3) is closed, its spectrum is discrete, namely, it consists of isolated eigenvalues λ_k ($k = 1, 2, 3, \dots$) of finite multiplicity, with $\operatorname{Re} \lambda_k \rightarrow -\infty$ as $k \rightarrow \infty$ (see, for example, [19]).

Theorem 1. *Let all eigenvalues of the operator (4) have negative real parts. Then the solution $w = 0$ of the system (1)–(2) is asymptotically stable. If the operator (4) has an eigenvalue with a positive real part, then the solution $w = 0$ is unstable.*

This theorem is a consequence of a more general statement on Lyapunov stability conditions of solutions of parabolic equations by linear approximation (see, e. g., [21, pp. 113–118]).

Throughout the following, a square real matrix B will be called *stable* if all its eigenvalues have negative real parts.

3.3. On the diffusion instability of the reaction-diffusion system. The stability properties of the solution $w = 0$ of the system (1)–(2) and the solution $w = 0$ of the linear finite-dimensional system

$$\frac{dw}{dt} = A(\mu)w, \quad w \in R^N, \quad (5)$$

(i. e., the system without diffusion) may be different. Namely, it is possible that for some $\mu = \mu_0$ the solution $w = 0$ of (5) is stable, while the solution $w = 0$ of (1)–(2) is unstable. In other words, the presence of diffusion can change the nature of the stability of the zero solution. In such a situation, they talk about *diffusive instability* of the system (1)–(2). The concept of diffusion instability is crucial in the study of systems with diffusion (see, for example, [1, 4, 7, 8, 20, 22]).

In a narrower (and at the same time more common) formulation, the *diffusion instability* of the system (1)–(2) at $\mu = \mu_0$ is called the situation when:

- the matrix $A(\mu_0)$ is stable;
- the operator $S(\mu_0)$ has either one real eigenvalue $\lambda_0 \geq 0$, or a pair of complex-conjugate eigenvalues $\lambda_{1,2} = \tau \pm i\omega$, where $\tau \geq 0$ and $\omega > 0$;
- all other eigenvalues of the linear operator $S(\mu_0)$ have negative real parts.

3.4. The concepts of Turing, Andronov–Hopf, and Turing–Hopf bifurcations. We will use the concepts of Turing bifurcations, Andronov–Hopf bifurcations, and Turing–Hopf bifurcations (for a detailed discussion of the concepts of Turing bifurcations and Andronov–Hopf bifurcations, see [16]).

We will say that the value of $\mu = \mu_0$ is the *bifurcation point Turing-complete* systems (1)–(2) if:

- the matrix $A(\mu_0)$ is stable;
- the operator $S(\mu_0)$ has a simple eigenvalue of $\lambda = 0$, and the rest of its eigenvalues have negative real parts.

We will say that the value of $\mu = \mu_0$ is the *bifurcation point Andronov–Hopf* systems (1)–(2) if:

- the matrix $A(\mu_0)$ is stable;
- the operator $S(\mu_0)$ has a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, where $\omega_0 > 0$, and the rest of its eigenvalues have negative real parts.

We will say that the value of $\mu = \mu_0$ is the *bifurcation point Turing–Hopf* systems (1)–(2) if:

- the matrix $A(\mu_0)$ is stable;
- the operator $S(\mu_0)$ has a simple eigenvalue $\lambda = 0$ and a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, while its other eigenvalues have negative real parts.

REMARK 1. The requirement that the remaining eigenvalues of the operator $S(\mu_0)$ have negative real parts is due to the following circumstance. If the operator $S(\mu_0)$ has an eigenvalue with a positive real part, then the dissipative structures or self-waves that arise during bifurcation will be inherently unstable.

REMARK 2. There are various interpretations of the concepts of Turing, Andronov–Hopf and Turing–Hopf bifurcations in the literature. From the point of view of general nonlinear dynamics (see, for example, [23]) bifurcations in the neighborhood of the equilibrium point $w = 0$ of the system (1)–(2) are associated with the transition (when the system parameters are changed) of one or more eigenvalues of the operator (4) through the imaginary axis. In particular, it is not required that the matrix $A(\mu_0)$ be stable, and the bifurcation itself may be a consequence of the eigenvalues of the matrix $A(\mu)$ crossing the imaginary axis. Although the specified bifurcation is not directly related to the presence of a diffusion component in the system (1)–(2), the study of such bifurcations is also of great interest in the study of qualitative and quantitative rearrangements in reaction-diffusion systems.

REMARK 3. Turing and Andronov–Hopf bifurcations have codimension one, and Turing–Hopf bifurcation has codimension two. It is sufficient to have one scalar parameter in the system to effectively study codimension one bifurcation problems, and it is required to have at least two independent scalar parameters to study codimension two bifurcation problems. In this paper, when studying the Turing and Andronov–Hopf bifurcation problems, it will be assumed that the parameter μ in the system (1)–(2) is scalar, and when studying the Turing–Hopf bifurcation problem, it is two-dimensional.

3.5. On Turing and Andronov–Hopf bifurcation scenarios. The Turing bifurcation scenario is related to the fact that when the parameter μ crosses the value μ_0 in the system (1)–(2), nonzero spatially inhomogeneous equilibrium points $w = w(x, \mu)$ appear in the vicinity of the equilibrium point $w = 0$. Specifically, if μ_0 is a bifurcation point, Turing, then, according to the general theory of bifurcations (see, for example, [23]), there is usually a sequence $\mu_n \rightarrow \mu_0$ such that for each $\mu = \mu_n$, the system (1)–(2) has a spatially inhomogeneous equilibrium point $w = w_n(x)$ such that $\|w_n(x)\|_{W_2^2} \rightarrow 0$ as $n \rightarrow \infty$.

Accordingly, the Andronov–Hopf bifurcation scenario is related to the fact that when the parameter μ passes through the value μ_0 in the system (1)–(2), non-stationary periodic solutions $w = w(x, t, \mu)$ appear in the vicinity of the equilibrium point $w = 0$. Namely, if μ_0 is a bifurcation point, if the Andronov–Hopf bifurcation is present, then, as a rule, there is a sequence $\mu_n \rightarrow \mu_0$ such that at each $\mu = \mu_n$ the system (1)–(2) has a non-stationary T_n -periodic solution $w = w_n(x, t)$, while $\max_t \|w_n(x, t)\|_{W_2^2} \rightarrow 0$ and $T_n \rightarrow T_0 = \frac{2\pi}{\omega_0}$.

The study of these scenarios requires a detailed analysis of the system (1)–(2). In this regard, we note that the statements given in this article on the signs of Turing bifurcations and Andronov–Hopf system (1)–(2) are defined properties of only the linearized equation for $\mu = \mu_0$. They do not provide answers to questions about the details of the bifurcation scenario, and in particular, to the following important questions from a practical and theoretical

standpoint: at what values of the parameter μ do bifurcating solutions occur, what are the properties of their stability, and what is their geometric structure in the phase space of the system, among other things. Answers to these questions can be obtained by using the properties of the functions that are part of the system. (1)–(2), including the nonlinearity $h(w, \mu)$. The relevant issues are discussed, for example, in [16–18].

The Turing–Hopf bifurcation is characterized by significantly more complex dynamics than the classical Turing and Andronov–Hopf bifurcations. The Turing–Hopf bifurcation is a phenomenon in reaction-diffusion systems that involves the simultaneous birth of two parallel structures from a homogeneous state, unlike the Turing bifurcation or the Andronov–Hopf bifurcation, which involve the birth of a single structure. Turing–Hopf bifurcation makes two modes with different spatial periods unstable. This leads to spatially inhomogeneous structures, with one of them changing its position in time according to a periodic law. Turing–Hopf bifurcation includes stable and unstable equilibrium states, spatially homogeneous and inhomogeneous periodic solutions, and other complex spatio-temporal modes.

The Turing–Hopf bifurcation has codimension two. For its study and, in particular, for a detailed analysis of the scenarios of restructuring of the system when the parameters pass through the bifurcation points, methods of modern nonlinear dynamics can be used, in particular, methods of the theory of the central manifold, the theory of normal forms, the theory of multiparametric bifurcations (see, for example, [23–26]). In this regard, we note that in recent years, methods of normal form theory have been widely used in the study of complex bifurcations and, in particular, the Turing–Hopf bifurcation in reaction-diffusion systems (see, for example, [13, 14]).

4. Bifurcation Properties

4.1. Auxiliary matrices. Consider the family of matrices

$$B_k(\mu) = A(\mu) - k^2 D(\mu), \quad k = 0, 1, 2, \dots \quad (6)$$

The following statement holds (see, for example, [5]).

Theorem 2. *The set of eigenvalues of the operator (4) coincides with the set of eigenvalues of the matrices (6).*

It follows from Theorems 1 and 2 that if all matrices (6) are stable, then the equilibrium $w = 0$ will be an asymptotically stable solution to the system (1)–(2). If at least one of the matrices (6) has an eigenvalue with a positive real part, then the equilibrium point $w = 0$ will be unstable.

Theorem 2 also allows us to reformulate the above concepts of bifurcations. Turing, Andronov–Hopf, and Turing–Hopf in the following equivalent form.

The value of $\mu = \mu_0$ will be called the *bifurcation point Turing* systems (1)–(2) if:

T1) for some integer $k = k_0 \geq 1$, the matrix $B_0 = B_{k_0}(\mu_0)$ has a simple eigenvalue $\lambda = 0$, and the remaining eigenvalues of B_0 have negative real parts;

T2) the matrices $B_k(\mu_0)$ for $k \neq k_0$ are stable.

The value of $\mu = \mu_0$ will be called the *bifurcation point Andronov–Hopf* systems (1)–(2) if:

H1) for some integer $k = k_0 \geq 1$, the matrix $B_0 = B_{k_0}(\mu_0)$ has a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, and the remaining eigenvalues of the matrix B_0 have negative real parts;

H2) matrices $B_k(\mu_0)$ at $k \neq k_0$ are stable.

The value of $\mu = \mu_0$ will be called the *bifurcation point Turing–Hopf* systems (1)–(2) if:

TH1) for some integers $k = k_1 \geq 1$ and $k = k_2 \geq 1$, the matrix $B_1 = B_{k_1}(\mu_0)$ has a simple eigenvalue $\lambda = 0$, and the matrix $B_2 = B_{k_2}(\mu_0)$ has a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, while the remaining eigenvalues of the matrices B_1 and B_2 have negative real parts;

TH2) the matrices $B_k(\mu_0)$ when $k \neq k_1$ and $k \neq k_2$ are stable.

4.2. Auxiliary statements. We first note the following theorems.

Theorem 3. *The Turing bifurcation for the system (1)–(2) is possible only if $N \geq 2$.*

Theorem 4. *The Andronov–Hopf bifurcation for the system (1)–(2) is possible only when $N \geq 3$.*

Theorems 3 and 4 are well-known statements (see, for example, [4, 5]). The validity of these theorems also follows from the statements below, which are of independent interest.

REMARK 4. Theorems 3 and 4 hold under the assumption that the bifurcations considered in them are understood in accordance with the requirements of *T1–T2* and *H1–H2*. If bifurcations are understood in a broader sense (see Remark 2), then the Turing bifurcation for the system (1)–(2) becomes possible for any $N \geq 1$, and the Andronov–Hopf bifurcation — for $N \geq 2$.

4.3. Stability and bifurcation conditions. Let us discuss the bifurcation conditions and give the Lyapunov stability characteristics of the equilibrium point $w = 0$ of the system (1)–(2) for $N = 1$, $N = 2$ and $N = 3$.

Case $N = 1$. In this case, the equation (1) takes the form

$$\frac{dw}{dt} = \left(a(\mu) + d_1(\mu) \frac{\partial^2}{\partial x^2} \right) w + h(w, \mu), \quad w \in R^1,$$

where $a(\mu)$, $d_1(\mu)$ are smooth functions, with $d_1(\mu) > 0$. Accordingly, the matrices (6) become scalar functions:

$$b_k(\mu) = a(\mu) - k^2 d_1(\mu), \quad k = 0, 1, 2, \dots$$

Obviously,

Theorem 5. *Let the system (1)–(2) be one-dimensional, i. e., $N = 1$. Let*

$$a(\mu_0) - k^2 d_1(\mu_0) < 0, \quad k = 0, 1, 2, \dots \quad (7)$$

Then the equilibrium point $w = 0$ of the system (1)–(2) at $\mu = \mu_0$ will be asymptotically stable. If, for some integer $k = k_0$, the inequality $a(\mu_0) - k_0^2 d_1(\mu_0) > 0$ is satisfied, then the equilibrium point $w = 0$ is unstable.

To fulfill the conditions *T1* and *T2* of the Turing bifurcation, it is necessary to fulfill the inequality $a(\mu_0) < 0$. But then all inequalities will be fulfilled (7), i. e. the equilibrium point $w = 0$ will be stable. Accordingly, the conditions *T1* and *T2* of the Turing bifurcation are not satisfied. This, in particular, implies the validity of Theorem 1, equation (1) takes the form

$$\frac{dw}{dt} = \left(A(\mu) + D(\mu) \frac{\partial^2}{\partial x^2} \right) w + h(w, \mu), \quad w \in R^2,$$

in which

$$A(\mu) = \begin{bmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{bmatrix}, \quad D(\mu) = \begin{bmatrix} d_1(\mu) & 0 \\ 0 & d_2(\mu) \end{bmatrix};$$

here, for simplicity, we will assume that the matrix $D(\mu)$ is diagonal with positive elements $d_j(\mu)$.

Next, we use the fact that the stability of a second-order square matrix B is equivalent to the fulfillment of the inequalities $\det B > 0$, $\operatorname{tr} B < 0$. Applying this fact to the matrices (6) and taking into account the relations

$$\operatorname{tr} B_k(\mu_0) = \operatorname{tr} A(\mu_0) - k^2 \operatorname{tr} D(\mu_0), \quad \operatorname{tr} D(\mu_0) > 0, \quad (8)$$

we obtain that the following is true

Theorem 6. *Let the system (1)–(2) be two-dimensional, i. e., $N = 2$. Let*

$$\det A(\mu_0) > 0, \quad \operatorname{tr} A(\mu_0) < 0, \quad \det B_k(\mu_0) > 0, \quad k = 1, 2, 3, \dots \quad (9)$$

Then the equilibrium point $w = 0$ of the system (1)–(2) at $\mu = \mu_0$ will be asymptotically stable. If, instead of at least one of the inequalities (9) is the opposite strict inequality, then the equilibrium point $w = 0$ is unstable.

The Andronov–Hopf bifurcation is not realized here, since the conditions $H1$ and $H2$ require the simultaneous fulfillment of the inequality $\operatorname{tr} A(\mu_0) < 0$ and the equality $\operatorname{tr} B_{k_0}(\mu_0) = 0$ for some $k_0 \geq 1$, which is impossible.

To realize the Turing bifurcation, it is necessary that for some $k = k_0 \geq 1$ the following conditions were met:

$$\det A(\mu_0) > 0, \quad \operatorname{tr} A(\mu_0) < 0, \quad \det B_{k_0}(\mu_0) = 0, \quad \det B_k(\mu_0) > 0, \quad k \neq k_0. \quad (10)$$

Let us put

$$c_0 = a_{11}(\mu_0)d_2(\mu_0) + a_{22}(\mu_0)d_1(\mu_0), \quad \rho_0 = \frac{c_0}{d_1(\mu_0)d_2(\mu_0)} - k_0^2.$$

Theorem 7. *Let the system (1)–(2) be two-dimensional, i. e., $N = 2$. Let $c_0 \leq 0$. Then the Turing bifurcation in the system (1)–(2) at $\mu = \mu_0$ is not realized.*

The validity of this statement follows from the fact that, due to the equality

$$\det B_k(\mu_0) = \det A(\mu_0) - k^2 c_0 + k^4 d_1(\mu_0)d_2(\mu_0), \quad (11)$$

the necessary condition (10) for Turing bifurcation cannot be satisfied.

Thus, in the two-dimensional case, in order to implement the Turing bifurcation in the system (1)–(2), it is necessary that the inequality

$$c_0 = a_{11}(\mu_0)d_2(\mu_0) + a_{22}(\mu_0)d_1(\mu_0) > 0. \quad (12)$$

Note that since the second inequality in (10) must also be satisfied, the inequality $a_{11}(\mu_0) + a_{22}(\mu_0) < 0$ must also be satisfied. Therefore, the condition (12) cannot be satisfied if the values of $d_1(\mu_0)$ and $d_2(\mu_0)$ are not significantly different. In other words, in order to realize a Turing bifurcation, it is necessary that one of the diffusion coefficients $d_1(\mu_0)$ or $d_2(\mu_0)$ significantly surpasses the other. More accurate estimates for the values of $d_1(\mu_0)$ or $d_2(\mu_0)$ can be obtained, for example, based on the following statement.

Theorem 8. *Let $\operatorname{tr} A(\mu_0) = a_{11}(\mu_0) + a_{22}(\mu_0) < 0$ and $c_0 > 0$. Then the necessary condition (10) of Turing bifurcation is satisfied if and only if for some $k = k_0 \geq 1$, first, the inequality*

$$\det A(\mu_0) = -k_0^4 d_1(\mu_0)d_2(\mu_0) + k_0^2 c_0 > 0, \quad (13)$$

second, the interval $[\rho_0, k_0^2]$ (if $k_0^2 \geq \frac{c_0}{2d_1(\mu_0)d_2(\mu_0)}$) or the interval $[k_0^2, \rho_0]$ (if $k_0^2 < \frac{c_0}{2d_1(\mu_0)d_2(\mu_0)}$) does not contain integers of the form k^2 , $k^2 \neq k_0^2$.

Let us prove this statement. Necessity. Let conditions (10) be fulfilled. Then inequality (13) follows from the equality (11) at $k = k_0$ and conditions (10). The validity of the statement of the theorem about the interval $[\rho_0, k_0^2]$ (or $[k_0^2, \rho_0]$) can be established by contradiction. Namely, if this is not the case, then, for example, the interval $[\rho_0, k_0^2]$ will contain an integer k_1^2 , $k_1^2 \neq k_0^2$. But then we would have the inequality $\det B_{k_1}(\mu_0) \leq 0$, which contradicts the conditions (10).

Sufficiency. Let the inequality (13) and the statement of the theorem about the interval $[\rho_0, k_0^2]$ (or $[k_0^2, \rho_0]$) be fulfilled. It is enough to establish that the two final conditions in (10) are true. By the inequality (13) and the equality (11) at $k = k_0$ we obtain $\det B_{k_0}(\mu_0) = 0$. Further, from the statement of the theorem about the interval $[\rho_0, k_0^2]$ (or $[k_0^2, \rho_0]$) it follows that $\det B_k(\mu_0) > 0$ ($k \neq k_0$).

Case $N = 3$. In this case, equation (1) takes the form

$$\frac{dw}{dt} = \left(A(\mu) + D(\mu) \frac{\partial^2}{\partial x^2} \right) w + h(w, \mu), \quad w \in R^3, \quad (14)$$

in which

$$A(\mu) = \begin{bmatrix} a_{11}(\mu) & a_{12}(\mu) & a_{13}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) & a_{23}(\mu) \\ a_{31}(\mu) & a_{32}(\mu) & a_{33}(\mu) \end{bmatrix}, \quad D(\mu) = \begin{bmatrix} d_1(\mu) & 0 & 0 \\ 0 & d_2(\mu) & 0 \\ 0 & 0 & d_3(\mu) \end{bmatrix};$$

Next, we use the fact that the stability of a third-order square matrix B is equivalent to the fulfillment of the inequalities

$$\det B < 0, \quad \text{tr} B < 0, \quad b_0 = b_1 + b_2 + b_3 > 0, \quad b_0 \text{tr} B < \det B; \quad (15)$$

here b_1, b_2, b_3 are the principal minors of the matrix B . For matrices defined by equality (6) $B_k(\mu)$ define the numbers

$$a_k(\mu) = -\text{tr} B_k(\mu), \quad c_k(\mu) = -\det B_k(\mu), \quad b_k(\mu) = b_{1k}(\mu) + b_{2k}(\mu) + b_{3k}(\mu), \quad (16)$$

where $b_{jk}(\mu)$ are the principal minors of the matrix $B_k(\mu)$.

Applying the conditions (15) to the matrices (6) and taking into account the relations (8), we obtain that the following is true

Theorem 9. *Let the system (1)–(2) be three-dimensional, i. e., $N = 3$. For the stability of all matrices $B_k(\mu_0)$ (and, therefore, the asymptotic stability of the equilibrium point $w = 0$ of the system under consideration) it is necessary and sufficient that the inequalities*

$$\text{tr} A(\mu_0) < 0, \quad c_k(\mu_0) > 0, \quad a_k(\mu_0)b_k(\mu_0) > c_k(\mu_0), \quad k = 0, 1, 2, \dots \quad (17)$$

5. Signs of Bifurcations

5.1. Signs of Turing and Andronov–Hopf bifurcations. Let us first indicate the signs of Turing and Andronov–Hopf bifurcations in the three-dimensional system (1)–(2). From Theorem 9 it follows that the following statements are true.

Theorem 10. The value $\mu = \mu_0$ is a Turing bifurcation point of the three-dimensional system (1)–(2) if and only if:

– for some natural number k_0 , the following relations hold:

$$c_{k_0}(\mu_0) = 0, \quad a_{k_0}(\mu_0)b_{k_0}(\mu_0) > 0; \quad (18)$$

– relations (17) are fulfilled for all $k \neq k_0$.

Theorem 11. The value $\mu = \mu_0$ is an Andronov–Hopf bifurcation point of the three-dimensional system (1)–(2) if and only if:

– for some natural k_0 , the following relations are satisfied:

$$c_{k_0}(\mu_0) > 0, \quad a_{k_0}(\mu_0)b_{k_0}(\mu_0) = c_{k_0}(\mu_0); \quad (19)$$

– relations (17) are fulfilled for all $k \neq k_0$.

A simple analysis of the relations (18) and (19) shows that in order to implement both the Turing bifurcation and the Andronov–Hopf bifurcation, it is necessary that one of the diffusion coefficients d_1 , d_2 , or d_3 significantly exceeds the other two.

5.2. Signs of Turing–Hopf bifurcation. As noted above (see Remark 3), the problem of Turing–Hopf bifurcation is naturally studied under the assumption that the system (1)–(2) contains two parameters. Let’s assume that one of them is the scalar parameter μ , and the matrix A depends only on this parameter, i.e., $A = A(\mu)$. We will consider a small positive parameter ε as the second parameter, on which the diffusion matrix D depends so that

$$D = D(\varepsilon) = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (20)$$

The specified representation of the matrix $D(\varepsilon)$ corresponds to the above-mentioned properties of the coefficients of this matrix.

Thus, the system (1)–(2) in the case under consideration takes the form

$$\frac{dw}{dt} = \left(A(\mu) + D(\varepsilon) \frac{\partial^2}{\partial x^2} \right) w + h(w, \mu), \quad w \in R^3, \quad (21)$$

$$\frac{\partial w}{\partial x}(0, t) = \frac{\partial w}{\partial x}(\pi, t) = 0. \quad (22)$$

The matrices (6) here have the form

$$B_k(\mu, \varepsilon) = A(\mu) - k^2 D(\varepsilon), \quad k = 0, 1, 2, \dots \quad (23)$$

The above concept of a bifurcation point Turing–Hopf is transformed here to the following form. The value of (μ_0, ε_0) (here $\varepsilon_0 > 0$) of the parameters (μ, ε) of the system (21)–(22) will be called the *bifurcation point Turing–Hopf* if:

TH1) for some integers $k = k_1 \geq 1$ and $k = k_2 \geq 1$, the matrix $B_1 = B_{k_1}(\mu_0, \varepsilon_0)$ has a simple eigenvalue $\lambda = 0$, and the matrix $B_2 = B_{k_2}(\mu_0, \varepsilon_0)$ has a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, while the remaining eigenvalues of the matrices B_1 and B_2 have negative real parts;

TH2) the matrices $B_k(\mu_0, \varepsilon_0)$ for $k \neq k_1$ and $k \neq k_2$ are stable.

Let's discuss the issue of signs of bifurcation. Turing–Hopf in the system (21)–(22). For this purpose, along with (23) we will consider a family of matrices

$$B(x, \mu, \varepsilon) = A(\mu) + xD(\varepsilon), \quad x \leq 0. \quad (24)$$

When $x = -k^2$, the matrices (24) coincide with the matrices (23).

Let

$$\begin{aligned} a(x, \mu, \varepsilon) &= -\text{tr} B(x, \mu, \varepsilon), \quad c(x, \mu, \varepsilon) = -\det B(x, \mu, \varepsilon), \\ b(x, \mu, \varepsilon) &= b_1(x, \mu, \varepsilon) + b_2(x, \mu, \varepsilon) + b_3(x, \mu, \varepsilon), \\ f(x, \mu, \varepsilon) &= a(x, \mu, \varepsilon)b(x, \mu, \varepsilon) - c(x, \mu, \varepsilon); \end{aligned} \quad (25)$$

here $b_j(x, \mu, \varepsilon)$ — principal minors of the matrix $B(x, \mu, \varepsilon)$. The functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ correspond to the numbers (16).

Theorem 12. *The value (μ_0, ε_0) (here $\varepsilon_0 > 0$) is a Turing–Hopf bifurcation point of the three-dimensional system (21)–(22) if and only if:*

– for some natural numbers k_1 and k_2 , the following relations hold:

$$c(-k_1^2, \mu_0, \varepsilon_0) = 0, \quad a(-k_1^2, \mu_0, \varepsilon_0)b(-k_1^2, \mu_0, \varepsilon_0) > 0; \quad (26)$$

$$c(-k_2^2, \mu_0, \varepsilon_0) > 0, \quad f(-k_2^2, \mu_0, \varepsilon_0) = 0. \quad (27)$$

– relations (17) are fulfilled for all $k \neq k_1$ and $k \neq k_2$.

The validity of this statement follows from the fact that the relations (26) correspond to the Turing bifurcation with the wavenumber k_1 , and the relations (27) correspond to the bifurcation Andronov–Hopf wave number k_2 (see Theorems 10 and 11).

Note that if (μ_0, ε_0) is the Turing–Hopf bifurcation point of a three-dimensional system (21)–(22), then the wave numbers k_1 and k_2 corresponding to Turing bifurcation and bifurcation Andronov–Hopf, respectively, are different. This fact follows from equations (18) and (19), which show that the Turing bifurcation and the Andronov–Hopf bifurcation cannot occur at the same values of k_0 .

The signs of Turing–Hopf bifurcation proposed in Theorem 12 can be reduced to computational formulas that make it possible to effectively determine such bifurcations. In the task of developing such formulas, the main role is played by the study of the system of equations

$$\begin{cases} c(-k_1^2, \mu, \varepsilon) = 0, \\ f(-k_2^2, \mu, \varepsilon) = 0, \end{cases} \quad (28)$$

in which k_1 and k_2 are integer parameters. If this system is under certain natural conditions k_1 and k_2 have a solution of $\mu = \mu_0$ and $\varepsilon = \varepsilon_0 > 0$, then the basic condition of the TH1 Turing–Hopf bifurcation will be satisfied.

Let us give an example. Consider the system (21), in which $D(\varepsilon)$ is the matrix (20), and the matrix $A(\mu)$ has the form

$$A(\mu) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & \mu \\ -8 & -4 & -5 \end{bmatrix}.$$

Calculations show that system (28) with $k_1 = 16$ and $k_2 = 1$ has a solution $\mu = \mu_0 \approx 2.588$ and $\varepsilon = \varepsilon_0 \approx 0.009916$. Therefore, the TH1 Turing–Hopf bifurcation condition for the corresponding matrix $B_k(\mu_0, \varepsilon_0)$ is satisfied for $k_1 = 16$ and $k_2 = 1$. In this case:

- the matrix $B_{k_1}(\mu_0, \varepsilon_0)$ for $k_1 = 16$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -2.147$, $\lambda_3 = -260.93$;
- matrix $B_{k_2}(\mu_0, \varepsilon_0)$ at $k_2 = 1$ has eigenvalues $\lambda_{1,2} = \pm 1.201i$, $\lambda_3 = -3.02$.

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Received May 24, 2025

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Владикавказский математический журнал

2026, Том 28, Выпуск 3, С. 60–73

ЗАДАЧА О БИФУРКАЦИИ ТЬЮРИНГА — ХОПФА В ТРЕХМЕРНЫХ СИСТЕМАХ «РЕАКЦИЯ-ДИФФУЗИЯ»

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Аннотация. В статье обсуждаются вопросы исследования локальных бифуркаций в окрестностях пространственно однородных положений равновесия системы «реакция-диффузия» в ограниченной области с однородными краевыми условиями Неймана. Основные результаты касаются изучения задачи о взаимодействии бифуркаций Тьюринга и Андронова — Хопфа, приводящего к бифуркации Тьюринга — Хопфа. В такой бифуркации проявляются два типа диффузионных неустойчивостей, что может приводить к разнообразным эффектам, сопровождающимся возникновением сложных пространственно-временных режимов. С целью изучения основной задачи предлагаются новые расчетные формулы для исследования задач о бифуркациях Тьюринга и Андронова — Хопфа в одномерных, двумерных и трехмерных системах «реакция-диффузия». Полученные формулы представляют самостоятельный интерес. Проведено исследование условий, приводящих к бифуркации Тьюринга — Хопфа. Основным результатом является установление ее реализуемости в трехмерных системах «реакция-диффузия». Приводится

соответствующий пример. Обсуждаются также возможные направления развития предложенных результатов в задачах исследования диффузионной неустойчивости и соответствующих бифуркаций.

Ключевые слова: реакция-диффузия, положение равновесия, устойчивость, бифуркация, Тьюринг, Андронов — Хопф, Тьюринг — Хопф.

AMS Subject Classification: 37G10, 35K57.

Образец цитирования: *Vasenina N. A. and Yumagulov M. G.* The Turing–Hopf Bifurcation Problem in Three-Dimensional Reaction-Diffusion Systems // Владикавк. мат. журн.—2026.—Т. 28, № 3.—С. 60–73 (in English). DOI: 10.46698/f6031-1077-3134-o.